2. "The newer type of secondary mathematics" by John A. Swenson, Wadleigh High School, New York City.

3. "The unit in mathematics" by E. R. Breslich, University of Chicago.

At the luncheon between sessions, Professor Hyde acted as toastmaster, and Professor U. G. Mitchell, of the University of Kansas, spoke to the subject, "What we would teach."

At the session in the afternoon, the following papers were presented:

1. "A solution of the difference equation: $f(x+1)=\Gamma(x)\cdot f(x)$" by Miss Corinne Hattan, University of Kansas, by invitation.

2. "Use of hyper-space in analytic investigations" by Professor R. G. Smith, State Teachers College, Pittsburg, Kansas, by invitation.

3. "Applications of mathematics to stellar astronomy" by Professor R. W. Babcock, Kansas State Agricultural College.

Abstracts of these papers follow:

1. Miss Hattan defined a function $G(x)$ by the difference equation and the value of the function at one point. The zero points of the function were found. Weierstrass's factor theorem was used to obtain a general form of a function having these zero points. The function was then limited to a form satisfying the two conditions of the definition. This method expressed $G(x)$ as an infinite product. A simpler product was obtained when the independent variable was a positive integer. Some other forms and properties of the function were stated without proof, and a graph was drawn for real values of the independent variable.

2. The principal part of this paper was devoted to the use of the projective differential geometry of curves in hyper-space in the determination of a canonical form for ordinary linear homogeneous differential equations of order $n+1$; $n\geqq 4$.

3. The mathematics involved in the computations of stellar astronomy is very simple, and is thus in striking contrast to the work done by astronomers in locating Neptune and Pluto. The laws used are the quantitative expression of results of terrestrial laboratory experiments, or direct observations, such as the inverse square law of intensity of illumination, Kepler's third law modified by Newton, and Doppler's principle expressed in equation form. Combining such laws as these, masses of the planets, sun, and stars may be computed. From the parallax of nearby stars, trigonometrically computed, diameters of distant stars of the same class may be found, which lead to specific gravities as high as 50,000 in some cases. These extraordinary results, due to observation of spectral lines, have been checked within 5 per cent by theoretical results from the theory of relativity.

LUCY T. DOUGHERTY, *Secretary*

ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Fort Worth at Texas Christian University on January 31, 1931. During the morning session papers of a purely mathematical nature were presented, while during the afternoon session papers dealing with the problems of teaching mathematics were given and discussed. The morning session was presided over by Professor L. R. Ford, who also presided over the afternoon session until after the election of officers. Professor H. E. Bray then assumed charge of the meeting.

The Texas Section has a standing committee to consider problems connected with the teaching of mathematics in the colleges and secondary schools of the state and to make recommendations for the improvement of instruction in mathematics in the secondary schools. This committee, which is co-operating with the State Department of Education, was called together by the chairman, Professor F. W. Sparks of Texas Technological College, on the evening of January 30. Mr. H. F. Alves of the State Department of Education met with the committee. A number of questions pertaining to the mathematics curriculum of the secondary schools and to the preparation of teachers of mathematics in the secondary schools were discussed and recommendations were made. These recommendations were presented to the general meeting of the Section during the afternoon session.

Those attending the meeting were the guests of Texas Christian University at a dinner given in the University dining room on the evening of January 31. Professor Sherer acted as toastmaster and introduced Dean C. D. Hall of Texas Christian University, who in turn introduced the principal speaker of the evening, President E. O. Lovett of The Rice Institute.

Forty-five persons were at the morning session and approximately sixty were in attendance during the afternoon session, including the following thirty-one members of the Association: B. T. Adams, L. M. Blumenthal, H. E. Bray, Myrtle C. Brown, J. W. Cell, Savannah L. Cross, Alice C. Dean, Bertha K. Duncan, Nat Edmonson, Jr., H. J. Ettlinger, G. C. Evans, G. W. Evans, C. E. Ferguson, L. R. Ford, C. A. Gilley, J. W. Harrell, Deborah M. Hickey, C. M. Howard, E. O. Lovett, A. A. McSweeney, C. A. Murray, Lela Oxsheer, J. E. Redden, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer, F. W. Sparks, Elizabeth T. Stafford, Jennie Tate, R. S. Underwood.

The following officers were elected during the afternoon session: Chairman for one year, Professor H. E. Bray of The Rice Institute; Vice-chairman for one year, Professor C. R. Sherer of Texas Christian University; Secretary-Treasurer for five years, Nat Edmonson, Jr. of Texas Technological College.

The following program was presented:

I. *Morning Session*

1. "Universal quadratic null forms in five variables" by Professor R. S. Underwood, Texas Technological College.

2. "Isothermic harmonic functions" by E. F. Beckenbach of The Rice Institute, by invitation.

3. "A general Vandermondian determinant" by Dr. Elizabeth T. Stafford, Texas Technological College.

4. "Note on the double points of certain plane curves" by Dr. L. M. Blumenthal, The Rice Institute.

5. "A proof of a theorem of Popovici" by Professor H. E. Bray, The Rice Institute.

6. "An elementary treatment of the definite integral" by Professor H. J. Ettlinger, The University of Texas.

Abstracts of the papers follow:

1. In this paper the modifications of the theory of universal quadratic null forms in three variables necessary for a generalization to five variables were discussed.

2. In this paper properties of harmonic functions in three dimensions similar to those expressed by the Cauchy-Riemann equations for two dimensions were obtained.

3. Dr. Stafford gave in this paper a method of evaluating a general Vandermondian determinant which arose in her thesis for the doctorate at the University of Wisconsin.

4. In this paper theorems analogous to those obtained by Halphen (Bull. de la Société Math. de France, 1882, p. 162) for the plane sextic are deduced for all curves of order $2n$ represented by a web of curves $\rho^2+2\sigma\rho\tau+p\tau^2+2r\rho=0$ where σ, τ are of degree n and ρ of degree $2n$. The general theory is applied in detail to the case of the octavic, where it is shown that if two quadruple points and seven double points are assigned arbitrarily, the locus of other possible double points is a curve of order 13 with seven-fold points at the two assigned double points and triple points at the double points of the octavic. It is seen that the locus is, then, composite, consisting of a line and a curve of order twelve. Without the aid of Cremona transformations of which no use is made in this paper, it is shown that in the pencil of octavics with two assigned quadruple points, seven arbitrarily assigned double points, and one double point assigned on the nodal locus, there are not more than 47 octavics with a ninth double point.

5. The theorem is as follows: If $P(x)$ is a polynomial whose roots, $x_1, x_2, \cdots x_n$, are all real and non-negative; if the roots of the derivative $P'(x)$ are $y_1, y_2, \cdots y_n$; and if m is a positive integer, then $\sum_n^1 x_i^m/n \geqq \sum_1^{n-1} y_i^m/(n-1)$. The equality holds if $m=1$ or if the roots, x_i, are all equal, but in no other case. This is a natural generalization of a well known property of polynomials (corresponding to the case $m=1$), and is interesting even with the restriction to the real domain. The method of proof consists in studying the variations in the roots of $P'(x)$ when the smallest root of $P(x)$ is held fast and the other roots are increased by the same amount h. The writer is indebted to Dr. Jacques Shohat for communi-

cating to him the statement of the theorem, which was originally proposed, without proof, by M. Popovici.

6. Professor Ettlinger outlined a treatment of the definite integral designed to be within the grasp of the student taking a course in advanced calculus and at the same time to be fairly rigorous. The idea of uniform continuity is used.

II. *Afternoon Session*

1. "The place of mathematics in engineering education" by W. J. Miller, Dean of Engineering, Texas Technological College, by invitation.

2. "A modern method of teaching freshman mathematics" by Professor C. R. Sherer, Texas Christian University.

3. Report of the committee of the Texas Section.

4. "Problems in connection with the teaching of high school algebra" by H. F. Alves of the Texas State Department of Education.

Abstracts of the papers follow:

1. The value of mathematics as a tool and as a discipline for the engineering student were discussed at length. Dean Miller stressed the fact that numerous problems of design in engineering are dependent on the work of mathematicians for their solution.

2. Professor Sherer described in this paper a method of teaching freshman mathematics in use in Texas Christian University. According to this method the students are separated according to initiative and ability by means of special examinations. In the upper division each student works on his own initiative and conferences with the instructor replace formal classroom work.

3. The committee described above presented at this time its recommendations relative to the mathematics curriculum in the secondary schools and to the preparation of teachers of mathematics in secondary schools.

4. This paper was presented in connection with the work of the Committee of the Texas Section. In it Mr. Alves described the results of experiments conducted under his supervision on the teaching of high school algebra. In these experiments both the content and the method of teaching of the course in algebra were varied. Interesting and apparently favorable results were obtained.

NAT EDMONSON, JR., *Secretary*

SIXTEENTH ANNUAL MEETING OF THE OHIO SECTION

The sixteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 2, 1931, the day preceding the meetings of the Ohio College Association, with an afternoon session, dinner, and an evening session. The chairman of the Section, Professor W. G. Simon, presided at the afternoon session, which was devoted to the reading of papers. The evening session was given over to the con-

sideration of the report of a committee which is studying the mathematical situation in the high schools, and was in charge of the chairman of this committee, Professor I. A. Barnett.

Fifty-one persons registered attendance, forty-three of whom were members of the Association, namely: H. H. Alden, R. B. Allen, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, L. T. Black, M. G. Boyce, C. T. Bumer, R. S. Burington, F. E. Carr, E. H. Clarke, Rufus Crane, C. W. Dancer, D. S. Dearman, O. L. Dustheimer, T. M. Focke, B. C. Glover, Harris Hancock, R. C. Hildner, Margaret E. Jones, L. C. Knight, H. W. Kuhn, Lincoln LaPaz, F. A. Lewis, C. C. MacDuffee, Florentina Mathias, H. E. Menke, C. C. Morris, J. R. Overman, Tibor Radò, S. E. Rasor, C. E. Rhodes, Hortense Rickard, S. A. Rowland, W. G. Simon, H. E. Stelson, J. H. Weaver, R. B. Wildermuth, C. O. Williamson, B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, W. D. Cairns; Secretary-Treasurer, Rufus Crane; Member of executive committee, H. M. Beatty; Member of program committee, B. F. Yanney.

It is expected that the next meeting will be held at the Ohio State University on Thursday, April 7, 1932.

The following six papers were presented:

1. "Some doubts about the content of elementary courses in Calculus" by the chairman of the Section, Professor W. G. Simon, Western Reserve University.

2. "Linear differential equations" by Professor I. A. Barnett, University of Cincinnati.

3. "Small sample theory" by Professor C. C. Morris, Ohio State University.

4. "University publications and university presses" by Professor Harris Hancock, University of Cincinnati.

5. "Some algebraic limits and their geometric interpretation" by Professor E. H. Clarke, Hiram College.

6. "The problem of Plateau" by Professor Tibor Radò, Ohio State University.

Abstracts of these papers follow:

1. In view of the dissatisfaction with the work in mathematics done by the secondary schools, Professor Simon feels that college teachers should examine carefully the courses in college mathematics. He feels that the secondary schools have emphasized technique unduly; and after examining recent texts in calculus, he fears that students who take a first course in calculus acquire little more than a knowledge of its technique. He feels that the course should be reconstructed in such a way that more stress will be laid on fundamental principles.

2. It was shown in this paper that there exists a transformation of the Volterra kind which takes any linear homogeneous differential of the second order into one of a similar type. The formulas for the kernel of the Volterra transfor-

mation were derived in such a way that the resolvents of a certain class of kernels may readily be computed.

3. Small sample theory had its origin in a paper by Student which appeared in Vol. VI of *Biometrika* in 1908. Beginning with this paper, Professor Morris traced the development of the theory up to the present time. Solutions of typical examples served to show the difference in results obtained when formulas from the old theory and the new, were used. Special emphasis was placed upon Fisher's use of n-dimensional geometry in generalizing Student's results. The new statistical term, "likelihood," was defined and examples given of its use. Attention was called to the fact that the making of quality control charts, based upon small sample theory, marks a new epoch in industrial management.

4. For convenience Professor Hancock makes the first period of American mathematical publications begin with Nathaniel Bowditch's translation of Laplace's *Mécanique Céleste* in 1829–1839 and end with Sylvester's appearance at Johns Hopkins University in 1876. His second period extends up to the beginning of the present century. Roughly speaking, after enumerating important publications, Professor Hancock estimates that twenty-five thousand dollars would have paid for the publications of the first period which covers fifty years, and that a like sum would cover the expenses of publication of the second period. The first decade of the present century would have required at least that amount; while double that sum could have been profitably used for publications during the years 1910–1920; and twenty-five thousand dollars would be added to that amount for the decade just past. Judging from the increasing interest in mathematical studies, the rapidly growing number of universities that are offering courses in this subject, and the extraordinary number of papers that are being offered at the various meetings of mathematicians, it is conservative to say that at least two hundred thousand dollars could be profitably spent in mathematical publications for the next decade, 1930–1940. This sum is exclusive of mathematical journals. Unless something unforeseen happens this sum should be increased from decade to decade. The first question is: How should this sum be raised? Professor Hancock holds, and gives convincing argument for his contention, that the universities should contribute it, especially those universities whose buildings programs are practically completed. The next question is: How is it best to publish these mathematical works, through a few centralized presses like the Cambridge University Press of England, or through certain of our best American commercial publishing companies? Professor Hancock regards the two questions here raised as vital in the growth and promotion of American mathematics, and of such importance that they should be further considered by the mathematicians of the country.

5. Consider the ordinary plane analytic geometry problem of finding the mid-points of the sides of a triangle with permutable coefficients for the vertices. Continue the process to a second, third, etc., set of successive mid-points. The limiting point is the centroid. The formula for the resulting coordinate is in the binomial form and by equating coefficients a simple expression can be found for

the limit of the sum formed by taking the third, sixth, ninth, etc., binomial coefficient. This may readily be extended from three points to any number.

6. The object of this paper was to characterize, from the point of view of the calculus of variations, some of the classic geometrical extremal problems, such as the problem of the shortest line between two points, the problem of geodesics, and the problem of Plateau. The last problem was considered with more details, and the recent developments on this problem were discussed as sources of new type problems for the *calculus of variations*.

RUFUS CRANE, *Secretary*

ON THE ORIGIN OF THE HINDU TERMS FOR "ROOT"

By BIBHUTIBHUSAN DATTA, University of Calcutta

In a previous issue in this Monthly,[1] the writer treated of the origin and significance of two Sanskrit terms, *mûla* and *pada*, which are generally employed in Hindu mathematics to denote the "root" of a number. It is now found that this treatment requires further amplification and elucidation, especially in the light of later investigations. There is also another need of the present article. In the previous article, the writer ventured to criticise and even to contradict certain statements of Dr. Solomon Gandz hinting at an Arabic influence in the conception of the Hindu term *mûla*. He showed conclusively that while there could be no possibility of such an influence in the origin of the *mûla*, there was, on the contrary, a greater likelihood of its influencing the conception of the corresponding Arabic term *jadhr*.[2] In accepting this view of the writer in modification of his earlier one, Dr. Gandz has, in a more recent article in this Monthly,[3] made a fresh conjecture that the source and origin of the Hindu *mûla*, together with the Greek, Hebrew, and Arabic technical terms for the "root" of a number, is to be found in the corresponding Egyptian term *ḳnbt*. The literal significance of the word *ḳnbt* has been stated to be "corner" or "angle." Since "corner" or "cornerstone" implies the "lowest part," "basis," "foundation," Dr. Gandz argues that it must be equivalent to the word *mûla*, which also carries the latter significance. The modernism in this ingenious interpretation seems to be too palpable to take it as worthy of any serious consideration. Apart from that, however, it can be shown easily that such a sweep-

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[1] Vol. 34 (1927), pp. 420–423.

[2] The oldest Arabic writer to employ the term *jadhr* was Muhammad ibn Mûsâ al-Khowârizmî who lived about the year 825, while the term *mûla* occurs in Hindu works dating from several centuries before. It is well known that the Arabs became acquainted with Hindu mathematics about the year 770, when several Hindu treatises on mathematics and astronomy were translated into Arabic. We know also from the testimony of Rabbi ben Ezra how the Arab scholars of that time made their arithmetical operations exactly according to the Hindu methods (Smith and Ginburg, *Rabbi ben Ezra and the Hindu-Arabic Problem*, this Monthly vol. 25 (1918), p. 103.

[3] *On the origin of the term "root"* (Second article), vol. 35 (1928), pp. 67–75.

ing generalisation will be wholly unjustifiable so far at least as the Hindu origin is concerned. Indeed there is not the faintest idea of "corner" or "angle" in the conception of the Hindu terms for the "root." It may also be pointed out that the metaphor of the "corner" or "cornerstone" being the "basis" or "foundation" is unknown in the Indian culture. Hence the ancient Hindus could not possibly have considered it in their search for a technical term for the "root."

It was pointed out in the writer's previous article (p. 421) that the use of the term *mûla* in connection with the theory of numbers is found in the works of Âryabhaṭa (499). Its use has recently been traced to a much more ancient work, the *Anuyogadvāra-sūtra*,[1] a canonical work written before the beginning of the Christian era. In this there occurs the terms *varga-mûla* ("the square root"), *varga-mûla-ghana* ["the cube of the square-root," meaning $(\sqrt{a})^3$]. In the same treatise there is also reference to higher roots. We next find the term in the *Tattvārthādhigama-sūtra-bhāṣya* (or "The Commentary on the Tattvārthādhigamasūtra") and the *Jambudvīpasamāsa* of Umāsvāti,[2] who lived at Kusamapura (ancient Pāṭalīputra, near modern Patna) about the middle of the second century of the Christian era, and probably much earlier.[3] Further if it be remembered that all these are religious works dealing primarily with logical, psychological, and metaphysical conceptions and the mythology of the early Jainas, the reference to the mathematical and other scientific matters being only incidental, it must be concluded that the term *mûla* was already very familiar by their time, and must have entered the theory of numbers long before. Thus it is proved conclusively that the use of the word *mûla* in the sense of the "root" of a number is much older than the corresponding use of the Hebrew word *iqqār*.[4]

Tracing the early history of the word *mûla*, we find that it was employed only once in the *Rg-veda*,[5] where it denotes the "foot" (of an *asura* or "demon"). It occurs twice in the *Vājasaneya Saṁhitā*,[6] and in each case it implies "the root of a plant." There are a several instances of its use in the *Atharvaveda*[7] in

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[1] *Anuyogadvāra-sūtra*, Sūtra 142. For further informations see the author's article on *The Jaina School of Mathematics*, in the Bulletin of the Calcutta Mathematical Society, vol. 21, p. 137ff.

[2] *Tattvārthādhigama-sûtra* of Umāsvāti with the *Bhāsya*, edited by K. P. Mody, Calcutta, 1903, iii, 11 (*com.*). *Jambudvīpasamāsa* of Umāsvāti appears as Appendix C of this edition. The application of the term *mûla* will be found in the 4th *Āhnika* of the work.

[3] There is a great divergence of opinions about the time of Umāsvāti. According to the tradition of the *Digambara* sect of the Jainas he lived in the year 714–798 after Srī Vīra (that is,135–219). But according to the *Svetāmbara* tradition, he lived *c.* 175 B.C. Satis ChandraVidyabhushan thinks that he flourished in the 1st century A.D.

[4] The term *iqqār* occurs in the oldest Hebrew geometry, *Mishnath ha-Middoth*, which according to Shapiro and Gandz, was written *c.* 200 A.D. But Smith suspects it to be a work of much later date, posterior probably to that of Al-Khowārīzmī (825).

[5] x, 87. 10.

[6] i, 25; xxii, 28.

[7] For instance compare: "root of a plant"—ii, 7. 3; iii, 23. 6; viii, 7. 2, 12; "root of a hair"—iii, 137. 3: "basis, foundation"— xiii, 1. 56, 57; xii. 9. 4.: "root cause"—vi, 44. 3; xii, 11. 2.

relation to the vegetable kingdom as also in the abstract sense of "lowest part," "basis," "foundation," "origin," or "cause." There is copious use of the word *mûla*, in its different meanings, throughout the other Vedic literatures including the *Brâhmaṇas* and the *Śrauta-sûtras*. It is noteworthy that in the *Śulba-sûtras* of Baudhâyana, Âpastamba, and Kâtyâyana, we do not find any application of the *mûla* in any sense.

The use of the word *pada* is more common than that of the word *mûla* in the ancient Sanskrit literatures of India. It will not be possible, nor is it necessary, to write here the complete history of the word *pada*. It will suffice for the purpose of the present article to state that even in the *Rg-veda*,[1] there are found numerous instances of its use in the sense of "foot," "footsteps," "places," "direction," or "source."

The earliest application of *pada* in a technical sense is found in the *Taittirîya Saṁhitâ*,[2] where it denotes a unit of measure of length employed in the construction of altars. In the *Śulba-sûtras* (the earliest of which was composed before 800 B.C.), the term *pada* implies a unit of linear measure as well as a unit of surface measure. The relation between these two kinds of units has been described by Âpastamba (*c.* 500 B.C.) thus: "By means of a unit of measure (of length) is generated a unit of measure (of surface)."[3] It implies that a unit of surface measure is the area of a small square each side of which measures a unit of length. This appears clearly from the instances given by Âpastamba in illustration of the above rule. He says: "(Here and generally elsewhere) a square (is implied); other figures (are to be meant) when expressly stated. By means of two (units of length as the side of a square is produced a surface containing) 4 (units of surface measure); by means of 3 (units of length are produced) 9 (units of area)."[4] Similarly we find in the *Kâtyâyana-Śulba-pariśiṣṭa*:[5] "The unit of measure (of area) is a square; it may be other figures when expressly stated. Two units (of length) generate four units (of area); three units (of length) generate nine units (of area); four units (of length) generate sixteen units (of area)." Thus here we have the most definite evidence of the application of the word *pada* in a concrete sense, as early as the time of the *Śulba-sûtras*.

The dimensions of the altars are specified sometimes by the number of *padas* (or other units of length) in their sides and sometimes by the number of *padas* (or other units of area) contained in them. Thus we have in the *Âpastamba Śulba-sûtra*,[6] "The western side is thirty *padas* or *prakramas* long, the *prâchî* or east line(i.e., the line drawn from the middle of the western side to the middle

[1] "Cause," "source," iii, 55. 14, 15; "foot," i, 154. 4; viii, 64. 2; "place," i, 67. 3, i, 164. 41; etc.

[2] vi, 2. 4. 5.

[3] *Âpastamba Śulva-sûtra*, iii. 4. This work has been edited by A. Bürk with a German translation and notes and comments, together with a valuable introduction, *ZDMG*, Bds. 55 and 56.

[4] *Ibid.*, iii, 5–6.

[5] iii, 5–8.

[6] v, 1; also compare G. Thibaut, "Sulvasûtras" reprinted from the *Journal of the Asiatic Society of Bengal*, 1875, p. 9.

of the eastern side of the *vedi*) is thirty-six *padas* or *prakramas* long; the eastern side, twenty-four; this is the tradition for *vedi* at the soma sacrifices." Here *pada* obviously implies the unit of length. Again we have "Twenty-eight less thousand *padas* (are in) the *Mahāvedi* . . .".[1] "The *Sautrāmanikī vedi* (contains) 324 *padas*."[2] In these two instances *pada* certainly refers to the square plots. The similar use of *pada* is found also in the *Bṛhat Saṁhitā* of Varāhamihira (505).[3] The specifications given for the *Uttara-vedi* are not quite free from ambiguity. "The *Uttara-vedi* has ten *padas*."[4] According to Dvāraknātha Yajvā, a commentator of the *Baudhāyana Śulba-sūtra* this may mean either that there are 10 *padas* on each side, hence altogether 100 *padas* in the *vedi*; or that there are only 10 *padas* in the *vedi*.[5] Now this latter interpretation is shared by Kātyāyana and the other view by Rudradatta, the commentator of the *Āpastamba Śrauta-sūtra*. Rudradatta has also quoted the authority of an ancient sage, Bharadvāja, in support of his contention. Both Âpastamba and Baudhāyana were undoubtedly of this opinion. For in the sūtra immediately following the former has referred to an alternative specification about the *Uttaravedi*;[6] "Some say that the front should be smaller" than the rear instead of there being 10 *padas* along each side. In the opinion of Baudhāyana also the shape of the *Uttaravedi* must be a square.[7] Hence in this instance we get a kind of naming a square *vedi*, as a matter of fact of any square plot, by the number of elementary squares that may be lying on each of its side and into which it is divided. This system of naming is found to have been more explicitly followed in the early Buddhist literature.

In the Buddhist *Dīgha Nikāya*[8] and *Vinaya Pitaka*[9] (c. 500 B.C.), there is reference to two varieties of chess. One was used to be played on boards of eight times eight squares and the other on boards of ten times ten squares. In the former case the board was called the *aṭṭapada* (Sanskrit *aṣṭapada*) and in the latter case the *daśapada*. In each case, the word *pada* undoubtedly denotes a "square plot," and the system of nomenclature is evidently based, as has also been pointed out by the commentator Buddhagosa, on the number of *padas* that lie on each side of the board. These games were in existence in India long before 500 B.C.[10] This system of nomenclature must consequently have been

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[1] v, 7.

[2] v, 9.

[3] Lii, 48, 50, 56.

[4] *Āpastamba Śrauta-sūtra*, vii, 3. 10; *Āpastamba Śulba-sūtra*, vi, 8; *Baudhāyana Śulba-sūtra*, i. 98.

[5] *The Paṇḍit*, old series, vol. x, 1875, p. 49.

[6] *Āpastamba Śrauta-sūtra*, vii, 3. 11.

[7] *Baudhāyana Śulba-sūtra*, i, 80.

[8] *Dīgha Nikāya*, ed. Rhys Davids and Carpenter, vol. I, p. 6; see also Rhys Davids's English translation of the same, *Dialogues of the Buddha* I, p. 9.

[9] *Vinaya Pitaka;* ed. H. Oldenberg, vol. II, p. 10; vol. III, p. 180.

[10] Rhys Davids & Steele, *The Pali-English Dictionary* on "Daśapada."

much older. In later times, however, especially in the *Śilpa-śāstras*,[1] plots of lands similarly divided, into smaller ones, are named according to the number of plots they contain. Thus we have such names as *catuḥṣaṣṭipada* (or 64-*pada*), *ekāśītipada* (or 81-*pada*) and *śatapada* (or 100-*pada*). In the *Bṛhat Saṁhitā*[2] we get the name *aṣṭāṣṭapada* (or eight-times eight-pada) for the first, clearly indicating that there were eight rows each consisting of eight smaller squares. Bharat Muni, an ancient authority on the *Śilpa-śāstras*, quoted by Bhaṭṭotpata,[3] has given schemes for dividing plots of triangular and circular shape into 64 and 81 smaller plots but has still retained the usual old names.

We shall now proceed to discuss the source and origin of another term which is inseparably connected with the *mûla* and *pada*. It is the *varga*. It has been stated in the previous article (p. 421) that the Hindu mathematicians do not attempt a formal definition of the term *mûla* (or *pada*); they simply take it as indicative of an operation reverse to that implied by the term *varga*. For this latter term, of course, they have always a clearly formulated definition. The source and origin of the term *varga* is, therefore, bound to throw much light on the origin of the term *mûla* (or *pada*). The word *varga* literally means "rows," or "troops" (of similar things). But in the mathematical treatises of the Hindus it ordinarily denotes the square power and also the square figure. How the word *varga* came to be used in that sense has been clearly indicated by Thibaut. "The origin of the term," says he, "is clearly to be sought for in the graphical representation of a square, which was divided in as many "vargas," or troops of small squares, as the side contained units of some measure. So the square drawn with a side of five *padas*' length could be divided into five *vargas*, each containing five small squares, the side of which was one *pada* long."[4] This explanation of the origin of the term *varga* is confirmed by a passage of the *Śulba-sūtras*: "As many measures (units of some measures) a cord contains, so many troops or rows (of small squares) it produces (when a square is drawn on it)."[5] Thus it is certain that the term *varga* has a purely concrete concept in its origin. So the reverse process which means to find one out of the *vargas*, or the "basis" or "cause" of the *vargas*, must carry nothing but a similar concrete concept.

Another Sanskrit term is sometimes used to denote the square power. It is *kṛti*, literally meaning "doing," "making," or "action." This term is very expressive, for it carries with it the idea of specific preformance,—probably graphical representation. Hence the term *Kṛti-mûla* means "the cause of action." It is noteworthy that Umāsvāti has once used the term *prakṛti* for the square root.[6] The Sanskrit word *prakṛti* means "cause" or "original source."

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[1] Binode Behari Dutt, *Town planning in Ancient India*, Calcutta, 1926.

[2] Lii. 55; also compare Lii, 42.

[3] Commentary on *Bṛhat Saṁhitā*, Lii, 56.

[4] Thibaut. *Śulba-sūtras*, p. 48.

[5] *Âpastamba Śulba-sūtra*, iii, 7; *Kātyāyana Śulba-pariśista*, iii, 9; Compare—Thibaut, *Śulba-sūtra* p. 48. Also compare *Bṛhat Saṁhitā*, Lii, 49, where the word *varga* denotes a group of small squares (*padas*) in the *Ekāśīti-pada*.

[6] *Tattvārthādhigama-sūtra*, iii, 11 (*Com.*).

The oldest Hindu term for the square root of a number was *karaṇī*. This term is still retained in the Hindu mathematics, but its application has been restricted. It is now used in connection with the square roots of those numbers, the root of which cannot be obtained exactly. Or, in other words, it now indicates a surd. But in the beginning it had a wider scope and was used to denote the square root of any number.[1] Umāsvāti (*c.* 150) treated the terms *mûla* and *karaṇī* as synonymous.[2] Even in the *Āpastamba Śulba-sûtra*,[3] we find the use of term *catuṣkaraṇī*, meaning $\sqrt{4}$. The origin of the term *karaṇī* is interesting. Literally it means the "making one" or "producing one." It seems to have been originally employed to denote the cord used for the measuring of a square.[4] It then meant the side of any square,[5] and is so called because it makes a square (*caturasra-karaṇī*).[6] Thus we see that the term *karaṇī* which in later mathematics has only the highly abstract idea of a surd number, was in the beginning a concrete concept. Hence the original term for the square root was a concrete concept. We also learn from it why when in later times there arose the need of a new term for the square root the Hindus chose the word *mûla* ("cause," "basis," or "foundation") to replace *karaṇī*.

From what has been stated above it will be sufficiently clear that there is not the slightest trace of the idea of the "corner" or "angle" in the origin and source of the Hindu terms for the "root" of a number. Even the word *mûla* or the word *pada* is not known to have ever been used in Sanskrit in that sense.[7] Hence the conjecture of Dr. Gandz, about the influence of the Egyptian term *knbt* (meaning literally "corner" or "angle") on the conception of the Hindu term *mūla* is wrong.

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[1] Thibaut rightly apprehended this to be the original meaning of *karaṇī* but he was somewhat doubtful. "*Karani* meant at first the side of any square, after that possibly the square root of any number. Possibly I say, for in reality the mathematical meaning of *karani* was restricted. It was not used to denote the square roots of those numbers, of which the root can be exactly obtained, but only of those the root of which does not come out exact . . ." (*Sulba-sûtra*, p. 48). As will appear from the instances cited below, his doubt was due to insufficient knowledge in this matter. It is also noteworthy that we do not find in the *Śulba-sûtras*, any other technical term for the root of a number, though the operation of extraction of roots was not uncommon in them.

[2] Compare *Tattvārthādhigama-sûtra bhāsya* (iii, 11)—"viskambha krterdaśagunasya *mûlam* vrttapariksepa" and *Jambudrīpasamāsa* (iv) "viskambhavargadasagunakaranī vrttapariksepa." There is also another like passage which contains the word *karanī*,—"vikkambhavagga dahaguna *karanī* vaṭṭa pariraya hoi," I cannot exactly locate it, but it must be from one of the mūlasûtras of the Jainas and hence very ancient.

[3] ii, 6.

[4] Thibaut, *Śulba-sûtras*.

[5] *Āpastamba Śulva-sûtra*, ii, 4, 5; xii, 1, 5, 9; xiii, 1, 11; xv, 10; xviii, 5. *Baudhāyana Śulba-sûtra*, i, 53–56, 57, 60. Thibaut, *Śulbat-sûtra*, pp. 78, 49.

[6] *Baudhāyana Śulba-sûtra*, i, 60. In the *Āpastamba Sulba-sûtra* occurs the term *pakṣakaraṇī* which literally means "The maker of the wing."

[7] In the *Rgveda*, we find though in rare cases, the application of the compounds *nava-pada* (i, 164. 41; viii, 76. 12) and *nava-srakti* (viii, 76. 12) in the same sense, meaning the "nine directions" of the heaven. Hence in these cases the words *pada* and *srakti* have equivalent significance. Though

CONSTRUCTION OF A RATIONAL CANONICAL FORM FOR A LINEAR TRANSFORMATION

By ALBERT A. BENNETT, Brown University

The existence and properties of a rational canonical form for a linear transformation have proved of interest to various writers[1] and have significance for applications to difference and differential equations. A characterization of a common canonical form is provided by each of these writers here mentioned. The linear transformation in each case involves only a finite number n of linearly independent variables, but there are an infinite number (∞^n) of possible "chains" as the term is used by Dickson. Even in this problem one may distinguish between an existence proof and a construction. Both Kowalewski and Dickson call upon the reader to take a chain of maximum possible length. It is true that no chain can have a length greater than the given n. On the other hand for a given linear transformation the actual maximum attainable length may be less than n. So far as existence is concerned there is no doubt that among the ∞^n chains there is at least one of maximal length. It would seem desirable however (and to the school of Brouwer and Weyl, necessary) to provide the reader with some actual method of recognizing when such a chain is obtained and of obtaining such a chain by a finite sequence of steps. This constructive feature is missing in their discussions. Lattès adopts a somewhat different procedure. Here the reader is to determine the polynomials with rational coefficients which are factors of a given polynomial. This suffers from the same constructive failure. While the problem is rational in the sense that the desired factors are rational the operation of determining such factors is at least not elementary. The reader may be unable to ascertain whether a given polynomial is or is not reducible in the domain of polynomials with rational coefficients.

It seems of interest that by altering the sequence of steps, involving otherwise slight modifications of Dickson's discussion and by a corresponding change in definition, an effective construction may be provided by which in a finite number of steps, each strictly rational (not involving factorization or the choice of an unknown element to satisfy given conditions), the same canonical form may be secured. It is this construction that is here provided. The notation employed will be essentially that of Dickson, and a portion of the language here used at the start will be quoted verbatim from this source.

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in Sanskrit the word *srakti* commonly denotes the "corner" or "angle," those instances can hardly be relied upon in support of the theory of Dr. Gandz. They are mentioned here for what they are worth.

[1] S. Lattès, Faculté des Sciences de l'Université de Toulouse Annales, (3) vol. 3 (1914), pp. 1–84 (with references to Burnside, Laguerre, Frobenius, etc.). Lattès considers only the case of a non-vanishing determinant.

G. Kowalewski, Leipzig, Berichte, vol. 69 (1917), pp. 325–335.

L. E. Dickson, *Modern Algebraic Theories* (1926), pp. 89–98.

Let L be any linear transformation

$$L: T_i = \sum_{j=1}^{n} a_{ij} t_j \qquad (i = 1, \cdots, n)$$

with coefficients in any field F.

Let x_1 be any homogeneous linear function of $t_1, \cdots, t_n$, whose coefficients belong to F and are not all zero. Let X_1 denote the corresponding function of $T_1, \cdots, T_n$. From x_1 we may generate an *absolute chain*,

$$(1) \qquad X_1 = x_2,\ X_2 = x_3, \cdots X_{a-1} = x_a,\ X_a = [x_1, \cdots, x_a],$$

where the bracket denotes a homogeneous linear function of $x_1, x_2, \cdots, x_a$ with coefficients in F. We shall call a the length of the chain, or say that the chain has a links, and shall refer to x_1 as the *leader* of the chain. If $X_1 = mx_1$ where m is in F, (but may be zero), the chain has one link only and $a = 1$. If $X_1 \neq mx_1$, we write x_2 for X_1. Then X_2 is formed, being the same function of $T_1, \cdots, T_n$, that x_2 is of $t_1, \cdots, t_n$. If X_2 is a linear function of x_1 and x_2 (perhaps identically zero), we have (1) for $a = 2$; in the contrary case we write x_3 for X_2 and have three linearly independent functions x_1, x_2, x_3. Proceeding in this manner each x_1 gives rise through L to a uniquely determined absolute chain of a linearly independent functions $x_1, x_2, \cdots, x_a$, with coefficients in F. It may be noted that the process of testing linear dependence involves only the evaluation of a numerical determinant and is strictly rational.

For the absolute chain (1) it will be convenient to indicate more explicitly the expression $X_a = [x_1, \cdots, x_a]$. The following notation may be used:

$$(2) \qquad X_a = [x_1, \cdots, x_a] = \sum_{i=1}^{a} a_i x_i.$$

Within the absolute chain no subscript greater than a is required. However one may define arbitrarily x_q for every natural number q, by the recursion relation,

$$(3) \qquad x_{q+1}(t_1, \cdots, t_n) = x_q(T_1, \cdots, T_n).$$

It may happen that for every q greater than a certain number, x_q vanishes identically.

The equation called the *characteristic condition* for the chain of x_1,

$$(4) \qquad s_{a+1} = \sum_{i=1}^{a} a_i s_i,$$

is satisfied not only for $s_1 = x_1$, but also (for each $q \geqq 1$) by $s_1 = x_q$, since the substitution L is linear. For example,

$$x_{a+2} = \sum_{i=1}^{a} a_i x_{i+1},$$

since this may be written as

$$L(X_a) = \sum_{i=1}^{a} a_i X_i,$$

which is a consequence of (2).

For $n > a$, let y_1 be any homogeneous linear function of the initial variables, t_i, with coefficients in F, which is linearly independent of $x_1, \cdots, x_a$. Let Y_1 denote the corresponding function of the variables T_i. We may then generate from Y_1 a *relative chain* of b links, relative to the absolute chain of x_1, expressed by

$$(5) \qquad Y_1 = y_2, \ Y_2 = y_3, \cdots Y_{b-1} = y_b, \ Y_b = [x_1, \cdots, x_a, y_1, \cdots, y_b],$$

where $x_1, \cdots, x_a, y_1, \cdots, y_b$ are linearly independent. Indeed if $Y_1 = [x_1, \cdots, x_a, y_1]$, we have (5) for the case $b = 1$. In the contrary case, write y_2 for Y_1. Then if $Y_2 = [x_1, \cdots, x_a, y_1, y_2]$ we have (5) for $b = 2$. In the contrary case write y_3 for Y_2, and so forth.

The term "relative chain" may be extended to apply to a chain relative to a given number of suitably related independent absolute chains as will be discussed presently.

An absolute chain,

$$(6) \quad Y_1 = y_2, \ Y_2 = y_3, \cdots, Y_{b-1} = y_b, \ Y_b = [y_1, \cdots, y_b] = \sum_{i=1}^{b} b_i y_i,$$

is said to be *canonically consequent* to an absolute chain,

$$(1) \quad X_1 = x_2, \ X_2 = x_3, \cdots, X_{a-1} = x_a, \ X_a = [x_1, \cdots, x_a] = \sum_{i=1}^{a} a_i x_i$$

if and only if

(i) the elements of the set $x_1, x_2, \cdots, x_a; y_1, y_2, \cdots, y_b$ are linearly independent, and also

(ii) y_1 satisfies the characteristic condition of the chain of x_1.

The final canonical form satisfies the following theorem.[1]

Theorem: *By the introduction of new variables which are linearly independent homogeneous linear functions of the initial variables t_i with coefficients in the field F, any linear transformation L with coefficients in F may be reduced to a canonical form defined by a sequence of absolute chains*

[1] In the other discussions the length of chain is made the defining feature.

$$(7)\qquad \begin{cases} X_1 = x_2,\ X_2 = x_3, \cdots, X_{a-1} = x_a,\ X_a = \sum_{i=1}^{a} a_i x_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_1 = y_2,\ Y_2 = y_3, \cdots, Y_{b-1} = y_b,\ Y_b = \sum_{i=1}^{b} b_i y_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_1 = z_2,\ Z_2 = z_3, \cdots, Z_{c-1} = z_c,\ Z_c = \sum_{i=1}^{c} c_i z_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

where $x_1, x_2, \cdots, x_a; \cdots; y_1, y_2, \cdots, y_b; \cdots; z_1, z_2, \cdots, z_c; \cdots$; *are linearly independent, and where each given chain of the sequence is canonically consequent to every preceding chain in the sequence.*

The method of procedure is a ruthless one in which if at any stage an absolute chain is obtained of length greater than some previously tentatively accepted chain, this previously obtained chain and all subsequent to it in the tentative system are rejected. Each step in the construction adds a new chain or replaces an old one by a longer chain. The total sum of the lengths of all the chains in a canonical system is at most n, the original number of arguments. The process must therefore terminate.

The process of choosing canonical variables in place of such original variables as are not in the final canonical system is one by which explicit linear combinations of variables, t_i, with coefficients in F are introduced. As each new linear combination is introduced, it takes the place of an original variable t_i, which is no longer needed in the system. The original variable when once dropped need never be taken up again, whether or not the new variable is eventually retained. At each stage we have n linearly independent arguments.

Suppose for the purpose of induction, we have at a given stage the following n linearly independent linear functions of t_i, with coefficients in F,

$$x_1, x_2, \cdots, x_a; \cdots; y_1, y_2, \cdots, y_b; \cdots; z_1, z_2, \cdots z_c; u, \cdots, w,$$

where $a+\cdots+b+\cdots+c=p$, and where the number of variables in the set, $u, \cdots, w$, is $n-p$. Suppose the first of these constitute the temporary tentative partial canonical system,

$$(8)\qquad \begin{cases} X_1 = x_2, \cdots, X_{a-1} = x_a,\ X_a = \sum_{i=1}^{a} a_i x_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_1 = y_2, \cdots, Y_{b-1} = y_b,\ Y_b = \sum_{i=1}^{b} b_i y_i, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_1 = z_2, \cdots, Z_{c-1} = z_c,\ Z_c = \sum_{i=1}^{c} c_i z_i, \end{cases}$$

and suppose also that each of the chains in this temporary partial sequence is canonically consequent to every preceding chain in the sequence. At the initial

stage one may merely assume that $p=0$, and that $u, \cdots, w$, are respectively $t_1, \cdots, t_n$. The p variables $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; \cdots; z_1, \cdots, z_c$, will be called the *classified*, and the $n-p$ variables, $u, \cdots, w$, the *unclassified* variables.

Take u and form its absolute chain. If this absolute chain exceeds in length that of any one of the tentative canonical chains, reject this shorter chain from the tentative canonical system, and relegate its variables to the unclassified set. Hence we may assume that no classified chain is shorter than the absolute chain of $u=u_1$. Form the relative chain of u_1 relative to all the classified variables. We then have

$$U_1 = u_2, \cdots, U_{h-1} = u_h, U_h = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; u_1, \cdots, u_h].$$

Three cases may be distinguished: (1) This relative chain may be an absolute chain, that is the brackets may not include any of the variables $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; \cdots; z_1, \cdots, z_c$ explicitly. (2) This relative chain may not be an absolute chain, (and hence will be necessarily of length less than that of the absolute chain of u_1) and also h may be such that $h<e\leqq b$, where y_e is the last element of the classified system to appear explicitly in the brackets. (3) This relative chain may not be an absolute chain but it may happen that $h\geqq e$.

In the first case, as already stated, there will be no absolute chain retained in the classified system whose length exceeds h links. In the second case, write $e=h+g$, $g>0$. Write

$$U_h = u_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + my_e + [u, \cdots, u_h], (m \neq 0).$$

Form the linear combination

$$v_1 = u_1 - my_g \ (m \neq 0).$$

Since L is linear, we may form the chain of v_1, in place of that of u_1 and obtain for every q,

$$v_{q+1} = L(v_q) = u_{q+1} - my_{g+q}.$$

By hypothesis $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_e; u_1, \cdots, u_h$, are linearly independent. Hence also, $x_1, \cdots, x_a; \cdots; y_1 \cdots, y_e; v_1, \cdots, v_h$, are linearly independent, since each v_i introduces a new u_i explicitly. Forming v_{h+1}, we have

$$v_{h+1} = u_{h+1} - my_e = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + [v_1 - my_g, v_2 - my_{g+1}, \cdots, v_h - my_{e-1}],$$

or, combining,

$$v_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + [v_1, \cdots, v_h].$$

Now drop the unclassified variable u_1 in favor of the variable v_1 which is again an explicit linear combination of the original variables t_i, with coefficients in F. But for v_1 the last explicitly appearing bracketed argument precedes y_e. By repetitions of this process we are led either to Case 1, or shall succeed in obtaining

an unclassified variable that may be called again, u_1, for which the previous final explicitly appearing set has been entirely removed from the brackets. If this process by repeated application does not provide a new absolute chain it can only be by the occurrence of Case 3.

For the third case let $h \geqq b$, and write

$$U_h = u_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + my_e + [u_1, \cdots, u_h] \ (m \neq 0, e \leqq h).$$

In this case we shall show that the absolute chain of u_1 necessarily exceeds in length that of y, so that this case also may be eliminated. Since by hypothesis $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_e; u_1, \cdots, u_h$ are linearly independent, the following are certainly linearly independent, namely, $u_1, u_2, \cdots, u_h, u_{h+1}, \cdots, u_{b+h-e+1}$. Indeed u_{h+1} introduces y_e for the first time and with non-vanishing coefficient m. Similarly u_{h+2} introduces y_{e+1} for the first time and with the same non-vanishing coefficient. Finally $u_{b+h-e+1}$ introduces y_b for the first time and with this same non-vanishing coefficient. Since $e \leqq h$, the subscript $b+h-e+1$ at least exceeds b, so that as desired the absolute chain of u_1 in this case exceeds in length the absolute chain of y_1. In accordance with the general directions, the chain of y_1 is then relegated to the unclassified set.

In every case therefore we are led to a new absolute chain. The previous temporary classified system is retained unless the new absolute chain exceeds in length some classified chain.

It remains only to show that the new absolute chain is canonically consequent to all the classified chains still retained or gives rise to an explicitly determined still longer chain. The process is not interrupted even if an absolute chain is obtained which requires the rejection of the entire set of hitherto tentatively adopted classified chains, for this only occurs when an absolute chain arises longer than any in the temporary classified system. A single chain and all other variables in the unclassified set is always a possible temporary classified system.

Suppose then for the sake of induction that u_1 generates an absolute chain of length not exceeding that of the chains generated by each of the classified variables, $x_1, \cdots, y_1, \cdots, z_1$, and suppose that the characteristic condition for z_1 is satisfied but that the latest of the characteristic conditions not satisfied by u_1 is that of y_1. Then by hypothesis,

$$u_{b+1} \neq \sum_{i=1}^{b} b_i u_i.$$

Form now the new linear combination $v_1 = y_1 + u_1$, and form the absolute chain of v_1. Since in particular $y_1, y_2, \cdots, y_b, u_1, \cdots, u_h$ are linearly independent and each subsequent u_q is a linear combination of u's alone, we conclude that $v_1, v_2, \cdots, v_b$, are surely linearly independent. If v_{b+1} is a linear combination of $v_1, \cdots, v_b$, say,

$$v_{b+1} = \sum_{i=1}^{b} m_i v_i$$

we have

$$v_{b+1} = y_{b+1} + u_{b+1} = \sum_{i=1}^{b} b_i y_i + [u_1, \cdots, \mu_h] = \sum_{i=1}^{b} m_i(y_i + u_i).$$

But this can only occur if $m_i = b_i$ $(i = 1, 2, \cdots, b)$ since the y's and u's are linearly independent. But this relation is impossible by the hypothesis that

$$u_{b+1} \neq \sum_{i=1}^{b} b_i u_i.$$

Hence the absolute chain of v_1 exceeds in length that of y_1 and is to be retained while relegating y_1 to the unclassified set.

By these successive reductions a canonical system at length survives. That the system obtained is indeed the invariant system used by Dickson, follows from the fact that an arbitrary linear combination of these canonical variables, must satisfy the first characteristic condition, since each individual term satisfies this homogeneous relation. Thus the absolute chain $x_1, x_2, \cdots, x_a$ eventually obtained as leading chain of the canonical system is of maximum possible length for any linear combination of these canonical variables. But the canonical variables constitute n linearly independent linear combinations of the original n variables, t_i. Hence this chain is of the maximum attainable length for any possible linear combination of the original variables, t_i, with coefficients in F. Similar remarks hold for any linear combination of the variables $y_1, y_2, \cdots, y_b; \cdots; z_1, \cdots, z_e; \cdots$. The construction has become possible by using the characteristic conditions rather than the maximal lengths of chains, as the determining conditions.

ON THE EQUILATERAL HYPERBOLA

By J. R. MUSSELMAN, Western Reserve University

Some time ago Professor P. Appell[1] proposed the following problem: *Let A_1, B_1, C_1, D_1 be four points on an equilateral hyperbola, the orthocenters of B_1 C_1 D_1, C_1 D_1 A_1, D_1 A_1 B_1, A_1 B_1 C_1 lie also on the curve. Thus from four points A_1, B_1, C_1, D_1 we derive four points A_2, B_2, C_2, D_2. This process can be continued indefinitely; under what conditions will A_n, B_n, C_n, D_n coincide entirely, or in part, with A_1, B_1, C_1, D_1?*

Appell suggested a method of attack which was used by Emch.[2] The problem was solved in other ways by Goormaghtigh[3] and Ser.[4] The purpose in bring-

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[1] Nouvelles Annales de Mathématiques, vol. 18 (1918), pp. 41–42.
[2] The American Mathematical Monthly, vol. 26 (1919), pp. 63–65.
[3] Nouvelles Annales de Mathématiques, vol. 18 (1918), pp. 445–448.
[4] Nouvelles Annales de Mathématiques, vol. 19 (1919), pp. 220–228

ing up the problem again is to discuss the case in which the points A_2, B_2, C_2, D_2 fall in other orders on A_1, B_1, C_1, D_1; and also to point out an interesting case in which some of the points are not real.

If we take the equation of the equilateral hyperbola $xy=c^2$ in a parametric form as

$$x = ct, \quad y = c/t \tag{1}$$

and let the parameters of A_1, B_1, C_1, D_1 be t_1, t_2, t_3, t_4 respectively, then the parameters of A_2, B_2, C_2, D_2 are $-t_1\sigma^{-1}, -t_2\sigma^{-1}, -t_3\sigma^{-1}, -t_4\sigma^{-1}$ respectively where $\sigma = t_1t_2t_3t_4$. The parameters of A_3, B_3, C_3, D_3 are $t_1\sigma^2, t_2\sigma^2, t_3\sigma^2, t_4\sigma^2$ and in general the parameters of $A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$ are

$$(-1)^n t_1\sigma^k, \quad (-1)^n t_2\sigma^k, \quad (-1)^n t_3\sigma^k, \quad (-1)^n t_4\sigma^k, \tag{2}$$

respectively, where $k = [(-3)^n - 1]/4$. Hence if A_{n+1} should coincide with A_1

$$\begin{gathered}(-1)^n t_1\sigma^k = t_1 \\ \text{or } (-1)^n\sigma^k = 1; \quad k = [(-3)^n - 1]/4.\end{gathered} \tag{3}$$

For real solutions we have the well-known cases: $n=1, \sigma=-1$, which states that the given four points A_1, B_1, C_1, D_1 must themselves be orthocentric; $n=2$, $\sigma=1$, which arises when the four points lie also on a circle.

Can A_2, B_2, C_2, D_2 coincide with B_1, A_1, D_1, C_1? This would require

$$-t_1\sigma^{-1} = t_2, \quad -t_2\sigma^{-1} = t_1, \quad -t_3\sigma^{-1} = t_4, \quad -t_4\sigma^{-1} = t_3, \tag{4}$$

which equations have the solution $t_2 = -t_1$, $t_4 = -t_3$ and $\sigma = 1$. Hence the given four points must lie on a circle with A_1 and B_1, C_1 and D_1 at opposite ends of two diameters of the hyperbola. The center of the circle is at the origin and the square of its radius is $c^2(t_1^2+t_3^2)$. The four given points form a rectangle whose area is $2c^2(t_1^2-t_3^2)$. Now the tangents at A_1 and B_1 meet in a point whose coordinates[1] are

$$2ct_1t_2/(t_1 + t_2), \quad 2c/(t_1 + t_2). \tag{5}$$

For the above particular choice of parameters for A_1, B_1, C_1, D_1, the four tangents at these points become two sets of parallel lines which form a rhombus whose area is $16\,c^2/(t_1^2-t_3^2)$. The product of the areas of the rectangle and rhombus is $32c^4$. Hence we have the theorem: *If A_2, B_2, C_2, D_2 coincide with B_1, A_1, D_1, C_1 the given four points lie on a circle with A_1 and B_1, C_1 and D_1 at opposite ends of two diameters of the hyperbola. Hence the product of the area of the rectangle which they form, and the area of the rhombus formed by the tangents to the hyperbola at these points, is a constant.*

Naturally the four points A_1, B_1, C_1, D_1 can be so chosen that A_2, B_2, C_2, D_2 will coincide with C_1, D_1, A_1, B_1, or with D_1, C_1, B_1, A_1. No other arrangements are possible for real solutions.

[1] Loney, *Coordinate Geometry*, p. 300.

For the next theorem it is necessary to use the fact that if A^1 be any point on an equilateral hyperbola, there are four circles through A^1 and the center of the hyperbola O which touch the hyperbola at points M_1, M_2, M_3, M_4 respectively.[1] These last four points lie on a circle through O, with center at A, the point on the hyperbola diametrically opposite to A^1.

If M_1^1 be the point diametrically opposite to M_1, then A^1, B^1, M_1, M_1^1 are an orthic set of points, B^1 is the further intersection of the circle, through A^1 and O, and tangent at M_1, with the equilateral hyperbola. Hence the line A^1B^1 is perpendicular to $M_1M_1^1$ and cuts OM_1^1 at K which is the midpoint of OM_1^1. Hence it is easy to construct a circle tangent at any point of an equilateral hyperbola. To construct a circle tangent at M, produce OM to M^1, erect a perpendicular bisector of OM^1 which cuts the hyperbola in two points. The circle through these two points and the center O will be tangent to the equilateral hyperbola at the point M.

Let us look further at these points M_1, M_2, M_3, M_4. If we call their parameters t_1, t_2, t_3, t_4 respectively; then A the center of the circle passing through them will have for its parameter $\sigma_1/2$, where $\sigma_1=t_1+t_2+t_3+t_4$. The parameter of A^1 will be $-\sigma_1/2$. Since there are four circles through A^1 tangent to the hyperbola at the points M_1, M_2, M_3, M_4 respectively, there will be four further points of intersection B_i^1 $(i=1, 2, 3, 4)$ whose parameters are $-2/t_i^2\sigma_1$ respectively. Now the coordinates[2] of the center of a circle passing through any four points of a hyperbola with parameters t_1, t_2, t_3, t_4 are $c\sigma_1/2$, $c\sigma_3/2$ where $\sigma_3=t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4$. But since A lies on the hyperbola, these coordinates must satisfy the equation of the hyperbola, hence

$$\sigma_1\sigma_3 = 4. \tag{6}$$

Since the circle also passes through O, the center of the hyperbola, if

$$\begin{aligned} \sigma_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4, \\ \sigma_2 &= 0. \end{aligned} \tag{7}$$

Hence the quartic equation, whose roots are the parameters of the four points M_i, can be written as

$$t^4 - \sigma_1t^3 - 4\sigma_1^{-1}t + 1 = 0. \tag{8}$$

The quartic equation whose roots are the parameters of the four points B_i^1 is

$$\sigma_1^4t^4 + 32\sigma_1t^3 - 24\sigma_1^2t^2 + 8\sigma_1^3t + 16 = 0. \tag{9}$$

Now this quartic (9) is nothing else than the Hessian of (8) as can be verified. Further if we ask for those two quartics of the pencil $(8)+\lambda(9)=0$ for which the invariant I_2 vanishes, we find they are precisely (8) and (9) themselves. Hence the four points M_i are an equianharmonic set; the same is true for the four points

[1] Lemaire, Nouvelles Annales de Mathématiques, (6), vol. 11 (1927), p. 63.

[2] Loney, *Coordinate Geometry*, p. 301.

B_i^1. So we have the theorem: *If we draw through any point of an equilateral hyperbola the four circles which touch the hyperbola, these four points of contact form an equianharmonic set, the four further points of intersection form a second equianharmonic set, which set is also the Hessian of the first set.*

In general the four points B_i^1 do not lie on a circle. But if $\sigma_1 = 2$, the points B_i^1 will lie on the circle whose equation is

$$(x + 2c)^2 + (y + 2c)^2 = 14c^2. \tag{10}$$

The equation of the circle on the points M_i, if $\sigma_1 = 2$ will be

$$(x - c)^2 + (y - c)^2 = 2c^2. \tag{11}$$

The points A and A^1 are now the vertices of the equilateral hyperbola, so the whole figure can easily be constructed. If we look at the six Jacobian points given by the Jacobian of the quartic (8) when $\sigma_1 = 2$, we find only two are real, namely the vertices; the other four lie on the circle whose equation is

$$(x - c)^2 + (y - c)^2 + 4c^2 = 0. \tag{12}$$

In the general case there are likewise four circles through A and O which are tangent to the curve at four points M_i^1 $(i = 1, 2, 3, 4)$, which points lie on a circle passing through O with center at A^1. There will likewise be four points B_i, the further intersections of these circles and the equilateral hyperbola. But these points are the diametrically opposite points of the sets M_i and B_i^1, so they present nothing new. However the four points M_i furnish an illustration of a set of four points whose orthocenters are M_i^1; and the orthocenters of M_i^1 again, are the original points M_i.

A NOTE ON THE CHARACTERISTIC DETERMINANT OF A MATRIX

By ALFRED K. MITCHELL, Yale University

Several authors[1] have pointed out that when a determinant is defined by means of the generalized Kronecker delta many of the properties of determinants follow almost immediately. It is the purpose of this note to show how the properties of the generalized Kronecker delta lend themselves to the derivation of the characteristic determinant of a matrix as a polynomial in λ.

The generalized Kronecker delta has k superscripts and k subscripts and is alternating both in superscripts and subscripts. It may be denoted by

$$\delta^{r_1 r_2 \cdots r_k}_{s_1 s_2 \cdots s_k}.$$

If the superscripts are distinct from each other and the subscripts are the same

[1] See F. D. Murnaghan, in this Monthly, Vol. 32 (1925), p. 233; and O. Veblen, *Cambridge Tracts*, No. 24.

set of numbers as the superscripts, the value of the symbol is $+1$ or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; in all other cases its value is zero. F. D. Murnaghan[2] has shown that the generalized Kronecker delta has the properties of a mixed tensor, but we are not concerned here with its tensor properties.

We shall use, however, the following relations[3] satisfied by the generalized Kronecker delta. These relations are easily verified by counting the number of terms which occur in the indicated summations.[4]

$$\delta^{\alpha r}_{\alpha s} = (n-1)\delta^{r}_{s}, \tag{1}$$

where $\delta^{r}_{s}=1$ if $r=s$; $\delta^{r}_{s}=0$ if $r\neq s$.

$$\delta^{r_1\cdots r_k\alpha_{k+1}\cdots\alpha_m}_{s_1\cdots s_k\alpha_{k+1}\cdots\alpha_m} = \{(n-k)!/(n-m)!\}\delta^{r_1\cdots r_k}_{s_1\cdots s_k} \tag{2}$$

$$\delta^{\alpha_1\cdots\alpha_k}_{\alpha_1\cdots\alpha_k} = \frac{n!}{(n-k)!}. \tag{3}$$

Remembering that the repeated indices are summation labels we see that the expression[5]

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}E^{\beta_1}_{\alpha_1}\cdots E^{\beta_n}_{\alpha_n}$$

will be by definition the n rowed determinant $|E^{r}_{s}|$. Furthermore it is readily seen that

$$I_1 = \delta^{\alpha}_{\beta}E^{\beta}_{\alpha} = E^{1}_{1} + E^{2}_{2} + E^{3}_{3} + \cdots$$

is the sum of the elements of the principle diagonal of the determinant $|E^{r}_{s}|$;

$$I_2 = \frac{1}{2!}\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}$$

is the sum of the two rowed principle minors;

$$I_3 = \frac{1}{3!}\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}E^{\beta_3}_{\alpha_3}$$

is the sum of the three rowed principle minors; etc.

Let us observe that when we have an expression of the type

$$(\delta^{\alpha_1}_{\beta_1}E^{\beta_1}_{\alpha_1} + \delta^{\alpha_2}_{\beta_2}E^{\beta_2}_{\alpha_2} + \cdots \delta^{\alpha r}_{\beta r}E^{\beta r}_{\alpha_r})$$

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[2] Bulletin of the American Mathematical Society, Vol. 31 (1925), p. 323.

[3] See O. Veblen, *Cambridge Tracts*, No. 24, p. 9.

[4] Repeated Greek letter indices will be used for summation labels, the summation being from 1 to n.

[5] See F. D. Murnaghan, in this Monthly, vol. 32 (1925), p. 233.

where, as above, the repeated indices are summation or dummy labels then obviously

$$(4) \qquad (\delta^{\alpha_1}_{\beta_1}E^{\beta_1}_{\alpha_1} + \delta^{\alpha_2}_{\beta_2}E^{\beta_2}_{\alpha_2} + \cdots \delta^{\alpha_r}_{\beta_r}E^{\beta_r}_{\alpha_r}) = r\delta^{\alpha}_{\beta}E^{\beta}_{\alpha}.$$

Now by the characteristic determinant of a square matrix $\|E^r_s\|$ is meant the determinant of the matrix $\|\delta^r_s\lambda - E^r_s\|$. Thus the characteristic determinant of the matrix $\|E^r_s\|$ will be

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}(\delta^{\beta_1}_{\alpha_1}\lambda - E^{\beta_1}_{\alpha_1})(\delta^{\beta_2}_{\alpha_2}\lambda - E^{\beta_2}_{\alpha_2})(\delta^{\beta_3}_{\alpha_3}\lambda - E^{\beta_3}_{\alpha_3}) \cdots (\delta^{\beta_n}_{\alpha_n}\lambda - E^{\beta_n}_{\alpha_n}).$$

From this we obtain, on multiplying the factors in parenthesis and collecting terms, using (4) above,

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\Big\{\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\cdots\delta^{\beta_n}_{\alpha_n}\lambda^n - n\delta^{\beta_2}_{\alpha_2}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}\lambda^{n-1} + \frac{n(n-1)}{2!}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}\lambda^{n-2}$$

$$+ \cdots (-1)^s\frac{n!}{s!(n-s)!}\delta^{\beta_{n-s}}_{\alpha_{n-s}}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}\cdots E^{\beta_{n-s}}_{\alpha_{n-s}}\lambda^{n-s} + \cdots (-1)^n E^{\beta_1}_{\alpha_1}\cdots E^{\beta_n}_{\alpha_n}\Big\}.$$

Remembering that $\delta^r_s = 1$ if $r = s$; $\delta^r_s = 0$ if $r \neq s$, and multiplying through by the factor $(1/n!)\ \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ and making use of (1), (2) and (3) and the definitions of I_1, I_2, etc., we obtain

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} - I_3\lambda^{n-3} + \cdots (-1)^s I_s\lambda^{n-s} + \cdots (-1)^n I_n,$$

the characteristic polynomial of the matrix $\|E^r_s\|$.

THE HESSIAN CONFIGURATION AND ITS RELATION TO THE GROUP OF ORDER 216

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and

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The group of order 216 was first discussed synthetically by Herman Wiener,[1] who made models of cubic surfaces. Unfortunately, however, this discussion appeared only in pamphlet form and is not generally available. For this reason the following treatment,[2] believed to be new, may be of interest.[3]

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[1] H. Wiener, *Die Einteilung der ebenen Kurven und Kegel dritter Ordnung*, Martin Schilling (1901).

[2] Suggested by Professor Virgil Snyder in a recent course in cubic curves at Cornell University.

[3] It was not until the present paper was in galley form that our attention was called to Professor R. M. Winger's article, *The Ternary Hesse Group and Its Invariants*, Univ. of Washington Publications in Mathematics, vol. 1, pp. 60–80. An excellent bibliography accompanies this article.

Using homogeneous coordinates, the equation of any non-singular plane cubic curve can be put in the Hesse form

$$(a) \qquad f \equiv x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0.$$

This curve has nine points of inflexion whose coordinates are

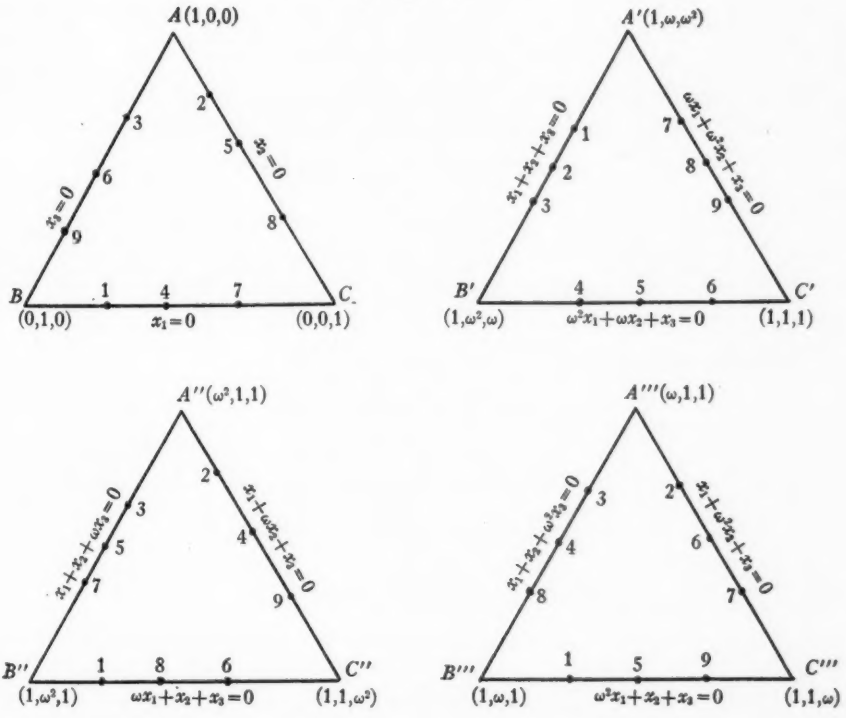
$$(0, 1, -1), \quad (1, 0, -1), \quad (-1, 1, 0),$$
$$(0, \omega^2, -1), \quad (\omega, 0, -1), \quad (-1, \omega, 0),$$
$$(0, \omega, -1), \quad (\omega^2, 0, -1), \quad (-1, \omega^2, 0),$$

where ω and ω^2 are the imaginary cube roots of unity.

Equation (a) contains one parameter m, and hence represents a pencil of cubics. Since the coordinates of the points of inflexion are independent of m, these points are points of inflexion for any curve of the pencil, called by Cayley a *syzygetic* pencil. For brevity we shall number these points and arrange them in the following square array:

(b)

1	2	3
4	5	6
7	8	9



It is easy to show that there are twelve lines, each of which contains three of these points of inflexion. These lines, called inflexional lines, are the lines containing the points in the horizontal, vertical, and diagonal rows of the above array. Each group forms a triangle, called an inflexional triangle. Any two of these triangles are in six-fold perspective; and any three are perspective from some vertex of the fourth. These triangles, with the coordinates of the vertices, the equations of the sides, and the points of inflexion which they contain are shown schematically in the accompanying figure. The notation is due to Clebsch[1] and the scheme is called the *Hessian Configuration*. This is the most famous configuration in the plane, with the possible exception of that of Desargues.

The polar conic of a point of inflexion is composite; it consists of the tangent at that point and a line called the harmonic polar of that point. Each harmonic polar contains a vertex from each of the triangles, but does not contain any of the points of inflexion. Denote by H_1 the homology which has (1) as center and its harmonic polar as axis, and by a similar notation the homologies which have the other points of inflexion as centers. Under H_1 the nine points of inflexion are permuted among themselves but in such a way that three points which are collinear remain collinear. Using the notation of groups we have, for example,

$$H_1 = (1)(23)(47)(59)(68),$$
$$H_2 = (13)(2)(49)(58)(67).$$

There are nine of these and it is obvious that H_i^2 is the identity.

Considering the product of any two of these transformations we find, for instance,

$$H_1H_2 = (132)(465)(798).$$

It is seen that the points group themselves by three's according to the rows of the original matrix (b). It is easy to verify that $(H_1H_2)^3$ is the identity. By constructing the multiplication table for the H's we find that the product of any two of them is equivalent to H_1H_i $(i=1, \cdots 9)$. Thus we have eight new operations of this type.

Next consider the product of three transformations, as

$$H_1H_2H_3 = (13)(2)(49)(58)(67).$$

But this is H_2. This illustrates the fact that if three centers are collinear, the product of the three corresponding homologies is the middle one. On the other hand the product $H_1H_2H_1$ is found to be H_3. That is, the product of three H's involving only two centers is the homology whose center is the third point collinear with the other two. If the centers are not collinear we have for example

$$H_1H_2H_4 = H_1H_1H_6 = H_1^2H_6 = H_6.$$

Indeed a product of the form $H_iH_jH_k$ can always be reduced to a single homology H_x.

[1] Clebsch-Lindemann, *Vorlesungen über Geometrie*, Bd. I, S. 507 (1876).

From the geometrical meaning of our transformation it is obvious that multiplication is associative, for instance,

$$(H_1H_2)H_3 = H_1(H_2H_3) = H_1H_2H_3,$$

and that the inverse is always the operation itself, two points being *interchanged* at each step.

Of the operations, H_i, H_iH_j, $(i=1 \cdots 9; j=1 \cdots 9)$ only 18 are distinct. These form a G_{18} called the Hessian Group.

The polar conic of the point (0, 1, −1) with respect to the cubic (a) reduces to

$$(x_2 - x_3)(x_2 + x_3 - 2mx_1) = 0,$$

where $x_2 - x_3 = 0$ is the harmonic polar of (0, 1, −1). Since under an homology lines through the center go into themselves and points on the axis are invariant, we can easily get the equations of H_1 by taking two lines through the center, and the new axis as the negative of the old, thus

$$x_1' = x_1;\ x_2' + x_3' = x_2 + x_3;\ x_2' - x_3' = x_3 - x_2.$$

Solving these equations gives

$$H_1;\ x_1' = x_1,\ x_2' = x_3,\ x_3' = x_2.$$

The equations for the other eight homologies can be found in a similar manner.

An example of one of the products is

$$H_1H_2;\ x_1' = x_2,\ x_2' = x_3,\ x_3' = x_1.$$

The transformations, I, H_1, H_2, H_3, H_1H_2, and H_2H_1 will be found to form a group. This is the symmetric group of permutations on three letters. These transformations are real. It follows that the Hessian Group has a sub-group of order six; these are the only transformations of the group that are real.

By applying any of these transformations it is found that $f=0$ is invariant. An ordinary point of $f=0$ is associated with 17 others, no three of the 17 being collinear. A point will not be invariant unless it lies on the harmonic polar of an inflexional point. A point of inflexion is transformed into a point of inflexion, and hence is associated with eight others. The behavior of the inflexional triangles will be considered later.

Now consider a transformation of the form

$$T_1:\ x_1' = \omega x_1,\ x_2' = x_2,\ x_3' = x_3,$$

of which there are three. There are nine combinations of these forming a group. The operations are commutative. Of these nine however, three are not new for $T_1^2T_2 = H_1H_4$, $T_1^2T_3 = H_1H_7$, and $T_1T_2T_3$ is the identity. These transformations are of period three. The six new ones represent homologies. For example T_1 and its inverse T_1^2 are homologies which have A for center and $x_1 = 0$ for axis.

The equations for these and further transformations are easily set up and will not be given.

If we multiply $H_1 \cdots H_9$ by T_1, T_2, T_3, we get transformations of period six. Multiplication in general is not commutative, and duplications occur. There are nine new transformations of this type. If we multiply the transformations H_iH_j by T_1, T_2, T_3, we get six new transformations of period three, making 15 in all. The products of these 15 by T_1^2, T_2^2, T_3^2 give 15 other new transformations which are the inverses of the original 15. We now have 54 transformations. These form a G_{54}, under which the triangle ABC is invariant. The other three triangles are permuted in various ways.

Turning now to the triangle $A'B'C'$, we find a similar situation with six new transformations T_i and T_i^2 $(i=4, 5, 6)$. These combine with the elements of the original G_{18}, to give 30 other new transformations, making in all a G_{54} of transformations under which this triangle is invariant as in the preceding case. A similar group exists for each of the other two triangles.

We now have 24 transformations of the form T_i or T_i^2. If we multiply any of these by any of the nine H_i we get transformations which are of period six. There are 72 of these. If we multiply T_i by H_jH_k we get 24 transformations of period three of which the products of T_i^2 by H_jH_k are the inverses. Thus we have 48 transformations of this type. Multiplication is not commutative.

If we combine the $T_i's$ among themselves we get nothing new. However, products of the form $T_i^2T_j$ $(i=1, 2, 3; j=4, 5 \cdots 12)$ do give something new. These and their inverses give 54 transformations of period four. The general product of type T_iT_j $(i=1, 2 \cdots 12; j=1, 2 \cdots 12)$ gives nothing further.

It is easy to show that the square of any of the products $T_i^2T_j(i=1, 2, 3; j=4, 5 \cdots 12)$ is one of the harmonic homologies and the product of any one of them by any of the G_{18} gives one of the 54. Hence G_{18} and these 54 form a group of order 72 of which G_{18} is an invariant sub-group.

We have now found 216 transformations which keep the points of inflexion of a cubic curve invariant, and these transformations form a group. Besides the identity there are six types:

(1) Nine harmonic homologies with the nine points of inflexion as centers. These are of period two. Each triangle is invariant, one side being fixed. On this side one point of inflexion is fixed and the other two points are interchanged.

(2) Eight transformations, each a product of two of type (1). Their period is three. Under any of these the sides of one triangle are invariant but the points of inflexion are permuted. The sides of the other triangles are permuted. The line joining the two centers is fixed.

The identity together with the transformations (1) and (2) form a G_{18}, the Hessian group.

(3) The 24 homologies having a vertex of an inflexional triangle for center and the opposite side for axis. They are of period three. In this case one triangle is invariant and the other three are permuted. The invariant triangle has one side invariant point by point, and the other two sides invariant as a whole.

(4) The 72 transformations formed by the product of (1) and (3). These are of period six. These homologies keep one triangle fixed and permute the others. The triangle that is invariant has one side invariant and the other two interchanged. This invariant side has one point fixed and the other two interchanged.

(5) The 48 transformations formed by the product of (3) and (2). These are of period three. One triangle is fixed with its sides permuted and the other three triangles are permuted.

(6) The 54 transformations formed from the products of the elements of (3). These are of period four, and leave no triangle invariant. On the other hand the triangles are permuted in pairs.

This G_{216} sends the syzygetic pencil $f=0$ into itself, each curve being sent into one of twelve (including itself). Each of these is invariant under a G_{18}. The images are as follows:

$$x_1^3 + x_2^3 + x_3^3 + Kx_1x_2x_3 = 0,$$

where K has the values

$$6m,\ 6m\omega,\ 6m\omega^2,\ \frac{6-6m}{1+2m},\ \frac{6\omega-6m\omega}{1+2m},\ \frac{6\omega^2-6m\omega^2}{1+2m},$$

$$\frac{6-6m\omega}{1+2m\omega},\ \frac{6\omega-6m\omega^2}{1+2m\omega},\ \frac{6\omega^2-6m}{1+2m\omega},\ \frac{6-6m\omega^2}{1+2m\omega^2},\ \frac{6\omega-6m}{1+2m\omega^2},\ \frac{6\omega^2-6m\omega}{1+2m\omega^2}.$$

We omit the details indicating which value of K corresponds to the separate transformations.

If $m=0$ the cubic is called *equianharmonic*. Under the G_{216} it goes into itself or one of the form

$$x_1^3 + x_2^3 + x_3^3 + 2ax_1x_2x_3 = 0,$$

where $a=1$, ω, or ω^2, and hence is invariant under a group of order 54. This curve has received more attention from the standpoint of group theory than any other cubic curve. The tangents at three collinear points of inflexion pass through the opposite vertex of the inflexional triangle. This is the only cubic curve whose inflexional triangles meet in a point.

The only other cubic curve which is invariant under a larger group than the Hessian is the *harmonic* cubic. The cross ratio of the four tangents to a cubic from an arbitrary point on the curve is constant and is called the characteristic of the curve. If this characteristic is -1 the curve is called harmonic. Such a curve is invariant under a cyclic, non-perspective transformation of period four whose square is the harmonic homology. The products of this transformation by those of G_{18} give 18 new ones, forming a group of order 36 of which the G_{18} is a sub-group. This curve then goes into one of six under the G_{216}.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

ON A TOTALLY DISCONTINUOUS FUNCTION

By ROBERT E. MORITZ, University of Washington

The following example, suggested by the Weierstrass function representing continuous curves which have no derivative at any point, seems to the writer unusually suggestive of the caution that must be employed in dealing with limits.

Consider the function

$$y(x, n) = f(x) + b^n \sin(a^n x\pi),\ 0 < b < 1, \tag{1}$$

together with its derivative

$$y'(x, n) = f'(x) + (ab)^n \pi \cos(a^n x\pi), \tag{2}$$

for increasing values of n, in any interval of x for which $f(x)$ is continuous.

We note that in such an interval both y and y' are continuous.

The graph of $y(x, n)$ results from the composition (addition of ordinates) for the two graphs of $y_1 = f(x)$ and $y_2 = b^n \sin(a^n x\pi)$ respectively, and consists, therefore, of a wave-curve superimposed on the curve $y = f(x)$. Both the wave-length and the amplitude of y_2 decrease indefinitely with increasing values of n. As n approaches ∞, $y(n, x)$ approaches $y = f(x)$.

Let us consider the nature of this approach more closely. While the wave-length $2/a^n$, and the amplitude b^n, of y_2 both decrease, its maximal slope (slope at a node) $(ab)^n\pi$ increases or decreases according as ab is greater or less than unity.

In the case $ab < 1$, the undulations of the curve y_2 not only grow shorter and lower with increasing values of n but they also become smoother and, in the limit, flatten out completely. In this case the tangent to $y(x, n)$ at any point in the interval assumes a definite direction as n approaches ∞. (2) shows that this direction is the direction of the tangent to $y = f(x)$.

In the case $ab > 1$, the undulations of the curve y_2 also grow shorter and lower with increasing values of n, but instead of becoming smoother, they become rougher, assuming a saw-tooth like shape; the teeth, though diminishing in size, grow sharper and steeper as n increases. In this case the tangent to $y(x, n)$ does not assume a definite direction as n approaches ∞. In fact, since $a^n x\pi$ changes by 2π while x changes by $2/a^n$, $(ab)^n\pi\cos(a^n x\pi)$ must take on all values from $-(ab)^n\pi$ to $+(ab)^n\pi$ while x changes by $2/a^n$, an interval which can be made less than any assignable quantity by taking n sufficiently large.

When $ab = 1$, the variations of the tangent to $y(x, n)$ for every interval of x however small are between $-\pi$ and $+\pi$.

This shows that $\lim y'(x, n)$ is totally discontinuous when $ab \geqq 1$, while when $ab < 1$, it is equal to the derivative of $\lim y$.

On Bell's Functional Equations

By Aaron Herschfeld, College of the City of New York

In the November, 1930 issue of this Monthly (vol. 37, p. 484) E. T. Bell requested the solution of the following two functional equations,

$$f(x, n_1)f(x, n_2) = f(x, n_1 + n_2 + c), \tag{3.1}$$

$$f(x, n_1)f(x, n_2) = f(x, cn_1n_2), \tag{3.2}$$

where n_1 and n_2 are integers $\geqq 0$ and c is a constant integer $\geqq 0$. A solution follows.

I. From (3.1) we have

$$f(x, n)f(x, 0) = f(x, n-1)f(x, 1) = f(x, n+c), \qquad n > 0.$$

$$\therefore f(x, n) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^n f(x, 0), \quad n \geqq 0, \quad f(x, 0) \neq 0,$$

while if $f(x, 0) = 0$ then by (3.1), $f(x, n) = 0$, $n \geqq 0$. Substituting the value of $f(x, n)$ in (3.1), we get, provided[1] $f(x, 1) \neq 0$,

$$f(x, 0) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^c,$$

so that

$$f(x, n) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^{n+c} = \{F(x)\}^{n+c},$$

$F(x)$ being an arbitrary function. This proves the generality of the solution given by Bell.

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[1] For $f(x, 1) = 0$, we have the single solution $c = 0, f(x, 0) = 1, f(x, n) = 0, n > 0$. Professor Franklin omitted this case in his solution of the first of the two functional equations. On page 154 of the March issue of this "Monthly" Franklin writes:

"$\cdots$; and also that if $f(x, 0) \neq 0$, no $f(x, n)$ can vanish."

The truth is that if $f(x, 0) \neq 0$, no $f(x, n)$ can vanish if $c > 0$, but if $c = 0$ and $f(x, 0) \neq 0$, then $f(x, n) = 0$, $n > 0$ and $f(x, 0) = 1$ is a solution. The following is a proof:

Case I. $c > 0$, $f(x, 0) \neq 0$. Using Professor Franklin's equations,

$$f(x, n-1)\, f(x, n+1) = f(x, 2n+c) = [f(x, n)]^2, \quad n = 1, 2, \cdots,$$

we see that should $f(x, n_1) = 0$ for some $n_1 > 0$, then $f(x, n) = 0$ for every $n > 0$. In that case $f(x, c) = 0$. But $[f(x, 0)]^2 = f(x, c)$, so that $f(x, 0) = 0$, which contradicts the hypothesis.

Case II. $c = 0$, $f(x, 0) \neq 0$. As before if any $f(x, n_1) = 0$, $n_1 > 0$, then $f(x, n) = 0$, $n > 0$. But now $[f(x, 0)]^2 = f(x, 0) \neq 0$. Hence $f(x, 0) = 1$. It is easily verified that the function $f(x, 0) = 1; f(x, n) = 0$, $n > 0$, actually satisfies Bell's equation.

II. From (3.2) we get

$$f(x, n_1)f(x, n_2) = f(x, 1)f(x, n_1n_2) = f(x, cn_1n_2). \tag{1}$$

Define the new function $\phi(x, n)$ by means of the relation, valid if $f(x, 1) \neq 0$,

$$\frac{f(x, n)}{f(x, 1)} = \phi(x, n).$$

Then

$$\phi(x, n_1)\phi(x, n_2) = \phi(x, n_1n_2). \tag{2}$$

For the case $f(x, 1) = 0$ we have, by (1), $f(x, n) = 0$, $n \geqq 0$.

Excluding the two trivial cases

$\phi(x, n) = 1$ for all $(n \geqq 0)$, which must hold if $\phi(x, 0) \neq 0$,

$\phi(x, n) = 0$ for all $(n \geqq 0)$, which must hold if $\phi(x, 1) \neq 1$,

the general solution of (2) is: $\phi(x, 0) = 0$; $\phi(x, 1) = 1$; for prime values of n, $\phi(x, 2)$, $\phi(x, 3), \cdots$, $\phi(x, p), \cdots$ are arbitrary functions of x; for any $n = p^aq^b \cdots$, where $p, q, \cdots$ are distinct primes,

$$\phi(x, n) = \{\phi(x, p)\}^a\{\phi(x, q)\}^b \cdots.$$

Hence, substituting $f(x, n) = f(x, 1)\phi(x, n)$ in (3.2), we get $f(x, 1) = \phi(x, c)$, so that $f(x, n) = \phi(x, nc)$. The particular choice $\phi(x, p) = p^{F(x)}$, yields the special solution $f(x, n) = (cn)^{F(x)}$, given by Bell.

A Note by the Editor

A solution of the same problem was given by Philip Franklin, in this Monthly, vol. 38 (1931), page 154. The present paper, without the footnote, was received while Franklin's article was in the press. R.E.G.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Numerical Mathematical Analysis. By James B. Scarborough. The Johns Hopkins Press, Baltimore, 1930. 416 pages. $5.50.

That there has been a remarkable development in the interest in numerical mathematics is shown by the numerous publications of this nature such as the valuable series of *Tracts for Computers* and other publications of the Biometric Laboratory headed by Karl Pearson, the publications of the Mathematical Laboratory of the University of Edinburgh, Running's *Empirical Formulas*,

the *Handbook of Mathematical Statistics*, edited by H. L. Rietz, Whittaker & Robinson's *Calculus of Observations*, Steffensen's *Interpolation*, to mention only a few of them. The subject of this review is a worthy addition to the group.

The author states in the preface that he has purposely refrained from employing symbolic methods and divided differences in deriving the standard formulas of interpolation. This is to be regretted, in my opinion, as not only is the derivation of these formulas much more direct, but all the common formulas are derived by a single method and these can be modified to suit the requirements of the special problem, where the ordinary formulas are inapplicable.

In Chapter I there is a valuable discussion on the accuracy of approximate calculations covering 37 pages. A valuable feature of the book is the excellent collection of examples at the end of each chapter. This feature would be improved if the answers were given, excepting in cases where the answer would spoil the problem. In Art. 10 is a discussion of the accuracy of series approximations with Taylor's formula (two forms) and Maclaurin's formula, with two forms for the remainder term for each series, one of them being in the form of an integral. This latter form is applied to an example: Find the remainder after n terms in the expansion of $\log_e(x+h)$. The result is that $R_n < (h/x)^n/n$ which is the first term of the series neglected—a well-known property of alternating series. It is possible, however, to obtain by elementary algebra a closer limit than this, viz.

$$R_n = h^n/[(x + h)n - h]x^{n-1}$$

As an example of the closeness of the approximation, let $h=1$, $x=1$; then $R_n = 1/(2n-1)$, with an error $< 1/4n^3$. Taking $n=10$, the error in log 2 is .00014. To get as close an approximation by taking the error as the first term neglected would require nearly 7000 terms of the series. In the example given by the author, $x=1$, $h=0.01$ and $n=4$, the formula just stated gives the remainder with an error only one-tenth that by the author's formula. On page 32 is given the logarithmic series

$$\log_e(m + 1) = \log m + 2[1/(2m + 1) + 1/3(2m + 1)^3 + \cdots].$$

The remainder term given is not as close as the following,

$$R_n = 1/[(4m^2 + 4m)n + 2m^2 + 2m + 1](2m + 1)^{2n-1}.$$

If we put $m=4$ and $n=5$, the series with the author's remainder term gives the value of log 5 to ten decimals. With the same number of terms of the series and the remainder term just given, the value of log 5 is correct to 14 decimals. Incidentally, there is a misprint at the top of page 32. In the first line, 5 should be 9. In the second line, 7 should be 5, and in the third the word "seven" should be replaced by "five."

Chapter II: *Interpolation. Differences. Newton's formulas for forward and backward interpolation.* The author gives the ordinary notation for differences

when arranged diagonally and another when arranged horizontally. It does not seem to the reviewer that the advantages in the new notation are great enough to compensate for the disadvantage of having a new notation to learn and use. If divided differences were used it would not be necessary to have two formulas, one for forward and the other for backward interpolation.

Chapter III: *Interpolation. Central-difference formulas.* The statement is made that Newton's interpolation formulas do not in general converge as rapidly as the central-difference formulas. It is doubtful if there is any appreciable difference in the rapidity of convergence in most cases. Stirling's and Bessel's interpolation formulas are derived by a long, involved method. Here would have been a good place to give Everett's central difference interpolation formula, which has many advantages over other formulas, one of them being that only half as many differences are required as with other formulas. It was for this reason that this formula was adopted in the *Logarithmetica Britannica*, the twenty-place table now being computed by Alexander J. Thompson and being issued by the Biometric Laboratory. (Four of the ten parts have been published.)

Chapter IV: *Lagrange's formula. Inverse interpolation.* Since the formulas derived in the preceding sections are applicable only when the values of the independent variable are given at equidistant intervals, it is important to have a formula which may be used when the values are given for any values of the independent variable. Such a formula is Lagrange's formula, which, however, was discovered by Euler originally. The author states that Lagrange's formula is tedious to apply and involves a great deal of computation. It has another disadvantage that if not enough data are used and it is desired to use more data, the whole computation must be gone over from the beginning. The author does not give any other method. Newton's *general interpolation* formula is the equivalent of Lagrange's formula and does not have the disadvantages of the latter. Although this formula is generally stated in terms of divided differences, it can be stated without them; see Boole's *Finite Differences*, 2nd edition, Chapter 3, Exercises 6 and 19. The problem of inverse interpolation may be solved by Lagrange's formula, but owing to the objections just stated, it is not commonly used. An alternative method is the method of successive approximations, or iteration, which is easily applied and has the advantage that any method of iteration possesses, which is that any errors are eliminated in the next trial, so that the result proves itself. Another method given is that of reversion of series. The coefficients in the reverted series are given as far as the 5th power of the variable and the method is applied to Newton's, Stirling's, and Bessel's formulas. The method is long and involves a great deal of computation with a good chance of errors. Newton's general interpolation formula could be used, of course.

Chapter V: *The accuracy of interpolation formulas.* The formulas for the remainder terms in the different interpolation formulas are given.

Chapter VI: *Interpolation with two independent variables. Trigonometric interpolation.*

Chapter VII: *Numerical differentiation and integration.* To find the derivatives of a function, it may be represented by an interpolation formula and this formula differentiated as many times as desired.

Numerical integration or mechanical quadrature is the process of computing the value of a definite integral from a set of numerical values of the integrand. By representing the function by Newton's interpolation formula, integrating, and substituting for the differences their values in terms of the ordinates, the formulas known as the Newton-Cotes formulas are obtained. Let the number of intervals be n. When $n=1$, we have the trapezoidal rule; when $n=2$, Simpson's one-third rule; when $n=3$, Simpson's three-eighths rule; when $n=6$, by modifying the coefficients slightly, we have Weddle's rule. The author observes that Simpson's three-eighths rule is less accurate than Simpson's one-third rule, although the reason is not stated. The reason is that if $n=2r$ the coefficient of the $(2r+1)$th difference vanishes so that the approximation is as good as or better than from the formula obtained by putting $n=2r+1$. This is the reason why the formulas obtained by putting $n=2, 4, 6, \cdots$ are more useful than those in which n is odd. Then the author's statement on page 120, line 5 that the geometric significance of Simpson's one-third rule is that we replace the graph of the given function by $n/2$ arcs of second-degree parabolas should be changed to "third-degree parabolas."

The remarks near the end of page 123 as to the relative accuracy of the different quadrature formulas admit of exceptions. For example in the integral

$$\int_{-1}^{1} dx/(e^x + 1)$$

the trapezoidal rule, Simpson's one-third rule, Simpson's three-eighths rule, and Weddle's rule all give the same value 1, which is the exact value. Merrifield, in his *Report on the Present State of Knowledge of the Application of Quadratures and Interpolation to Actual Data* (British Association Report, 1880), points out that in the rectification of the quarter-ellipse, quadrature formulas of high order do not give as good an approximation as those of lower order. If the semi-axes of the ellipse are 1 and $1/\sqrt{2}$, we have $y=\frac{1}{2}(3+\cos 2x)^{1/2}$, the limits for x being 0 and $\pi/2$. The correct value according to Legendre is 1.35064 38820. With 3 ordinates,

by the trapezoidal rule.....1.35055 with an error of 0.00009 0;

by Simpson's one-third rule...1.35381 8 with an error of 0.00317 4.

With 5 ordinates,

by the trapezoidal rule.....1.35064 3855 with an error of 0.00000 0027;

by Simpson's one-third rule...1.35088 3396 with an error of 0.000239;

by Cotes's formula $(n=4)$....1.35051 9660 with an error of 0.000124.

A good elementary presentation of Gauss's quadrature formulae is given. On page 135, the values of the u's and R's should be given in the surd form as well as in decimal form, as the former are frequently easier to substitute in a given problem. In Example 1, page 140, the value of an integral is found and the result is true to the 8th figure. However, if the calculations had been carried to more decimals, the value would have been found to be true to the 11th figure. Another important quadrature formula—not mentioned by the author—is Sheppard's Rule XI. (See the *Handbook of Mathematical Statistics*, edited by H. L. Rietz, page 18.)

In a great many of the computations, the author has not carried the work far enough to show the correct value of the error. An illustration of this is given in Example 1, page 121. By taking the logarithms to eleven places, the value given by Weddle's formula is 1.8278 47407 and the true value is 1.8278 47408. This gives the error by Weddle's formula of 1×10^{-9} instead of 3×10^{-8} as given by the author.

On page 142 the article "Caution in the use of quadrature formulas" the author warns against the use of quadrature formulae without ascertaining the nature and behavior of the integrand over the interval and illustrates with two examples. These are good illustrations but the author seems to have missed the reason for the lack of a good approximation to the value of the integral. The reason is stated by several authors, e.g. Merrifield (loc. cit.) and W. F. Sheppard in article "*Mensuration*" (Encyclopedia Britannica), §88, "Cases of Failure." Since the derivative in both examples becomes infinite at both limits, the integrand cannot be represented by a polynomial with integral exponents only. Several writers on the subject have derived special formulas for this case. However, a simple method which improves the accuracy of the formula is to consider a small portion of the curve adjoining the point where the derivative becomes infinite as a portion of a parabola with axis on the x-axis and vertex at the point mentioned. The area of such a segment bounded by a parabola is known to be two-thirds of the circumscribed rectangle.

In Example 1, page 142 there are some misprints. When $x=0.80$ the value of y is 0.038468. On page 144, line 4 from the bottom, instead of 0.000035, read 0.0000031. In line 2 from the bottom, instead of 0.023548, read 0.0206514. The same two corrections should be made in the last line. The result is given as 0.0235 and the inference is that all the figures are correct. However, if the number of ordinates is increased, the value is found to be 0.0238.

In Example 2, page 145, the author finds for the value of the elliptic integral for $h=0.1$ by Simpson's one-third rule 2.1934, the correct value being 2.2033. The error is 0.0099 or 0.45%. However, if we find the value of the integral between the limits -0.9 and 0.9 by Simpson's rule, we get 2.1226140. The area between -1 and -0.9 by the formula for the parabola, as suggested above, is $\frac{2}{3}(.1)0.7422\ 94=0.0494\ 863$. In the same way the area between $x=0.9$ and 1 is $\frac{2}{3}(.1)0.4571\ 65=0.0304\ 777$. Adding these three, gives for the value of the integral 2.2025780 with an error of only 0.0007 or 0.032% or less than one-tenth the error by using Simpson's rule alone.

A brief treatment of mechanical cubature or finding the approximate numerical value of a definite double integral is given with two examples worked out.

Chapter VIII is devoted to the accuracy of quadrature formulas. The relative accuracy of the different quadrature rules with formulas for the errors is discussed with illustrative examples.

Chapter IX is devoted to the solution of numerical algebraic and transcendental equations. Since Horner's method is explained in all college algebras, it is not discussed. The first method given is the method of false position. This is equivalent to using interpolation with first differences only, although it is not so stated. The Newton-Raphson method is next explained with a discussion of the inherent error. The method of iteration which is next given has many advantages which were mentioned in discussing the method applied to inverse interpolation. It requires only a slight knowledge of methematics and the errors are eliminated automatically. The last two methods are applied to simultaneous equations in several unknowns.

In Chapter X is a detailed explanation, covering 20 pages, of Graefe's root-squaring method for solving algebraic equations. There are a number of other methods of solving numerical equations which deserve at least a brief mention. In Workman's *Memoranda Mathematica* (Oxford, 1912) pp. 45–55 are given twelve methods. The graphical method is especially valuable when applied to cubic equations, as the roots may be found from the intersection of the curve $y=x^3$ and a straight line. This curve can be obtained in celluloid, making the graphic solution very rapid. Workman suggests that in Graeffe's method the series $f^3, f^9, f^{27}, \cdots$ is to be preferred to Graeffe's series for equations of degree not higher than 5. Another method for solving cubic equations is by trigonometry. (See Hobson's *Plane Trigonometry*.) Carl Runge in his *Praxis der Gleichungen* gives a method of calculating the root of any trinomial equation by addition and subtraction logarithms. In Thiele's *Interpolationsrechnung* is a solution of equations by divided differences, also interpolation of functions of two arguments.

Chapter XI treats the numerical solution of differential equations by the method of successive approximations; Chapter XII, the convergence and accuracy of the iteration process; and Chapter XIII, other methods for the numerical solution of differential equations, including the methods of J. C. Adams, Runge-Kutta, and W. E. Milne.

Chapter XIV: The normal law of errors and the principle of least squares.

Chapter XV: The precision of measurements.

Chapter XVI: Empirical formulas. The problem is to fit an equation to a set of numerical data. The methods used are: (1) The graphic method or method of selected points; (2) The method of averages; (3) The method of least squares.

The book has many admirable features. The explanations and derivations of formulae are given in detail and there are numerous examples at the end of each chapter. The author has avoided introducing new and complicated nota-

tions which, although they may conduce to brevity, are a serious stumbling block to the reader. The typography and paper are excellent and misprints are few.

The bibliographical references are very few. The book would be much improved by a good bibliography, so that those who wished a more extensive treatment of any topic would know where to go.

E. B. ESCOTT

A Manual of Greek Mathematics. By Sir Thomas L. Heath. Oxford University Press, 1931. xvi+552 pages. Price $5.00.

Although the author states in his Preface that "the story told in these pages is in substance the same as that unfolded in somewhat greater detail" in his *History of Greek Mathematics* (Clarendon Press, 2 vols., 1921), it is something much more than that. The main body of facts is naturally the same, but the work differs widely in purpose, somewhat in style, and naturally in various important details. In purpose it seeks to meet the needs and tastes of the general reader instead of the classical scholar; in style it cultivates somewhat more that which a recent essayist has called "a fetich of our time," namely, genuine simplicity,—the simplicity that appeals to the educated reader who is discouraged by any uncalled-for excess of erudition; and in details of knowledge it includes several discoveries which have been made since the History was published.

In the opening sentence the author seems to draw up a comfortable chair for his reader and to speak to him as an old friend,—"Why should we study Greek mathematics?" he asks, and this unassuming style is evident throughout the book. It is the style which the best teachers always seek, and it always appeals to the seeker after truth. And when the author answers the question he does so with a common-sense frankness that must appeal to even our most vociferous educational revolutionists, especially when he considers mathematics in the education of the Greek youth. For the Athenian boy of Plato's time, brought up under the latter's influence, was in great measure, a pupil of John Dewey's some two thousand years ahead of his time. Under the shadow of the Acropolis was born the project method, and here it was seen at its best. It is probable that our generation has often seen it at its worst. Pythagoras met and solved, for his day, the problems of "arrested development," "individual differences," the "I.Q.," and the College Entrance Examination Board,—but he thought and talked of something else.

These are, however, mere educational *obiter dicta* in this interesting summary of the story (for it is not a dry-as-dust "history") of the greatest intellectual development of ancient times,—probably of all time. In twenty-eight chapters the author runs the scale of Greek mathematics from the numbers of commerce and of calculation, through the remarkable development of geometry, to the algebra of Diophantus and the commentaries of the minor writers. This he does with a view to the needs of "the general reader who has not lost his interest in the studies of his youth and would wish to know how it came about that a Greek

of the name of Euclid wrote a text-book which, in an almost literal translation, was used in schools and in the universities in this country [Great Britain] as the one recognized basis of instruction in elementary geometry . . . until some fifty years ago. . . . The story of the creation of mathematics as a science by the Greeks and of the development of the Elements in the period ending with Euclid occupies about one-half of the present volume."

It should not be thought, however, that the work is merely a setting forth of praise for Euclid and his predecessors or as a condensation of the *History* mentioned above. It does not treat of Greek mathematics as an isolated structure of separate cubicles. "The elements did not spring fully formed from the brain of Euclid like Athene armed from the head of Zeus." This great teacher of Alexandria had his predecessors, and even Thales and Pythagoras had theirs, and what is probably the greatest question in the history of ancient mathematics at present is that relating to the possible precursors of the founder of the Ionian school at Miletus. For some time there have been advanced suggestions that the Greeks were not the founders of demonstrative geometry, but, until last year these were little more than mere opinions resting upon a single boastful phrase of Democritus (c. 400 B.C.). Now, however, evidence is appearing which gives ground for reconsidering the whole matter. The Moscow Papyrus, already known by name and slightly by content to readers of mathematical history, has at last appeared in translation (*Studien und Quellen zur Geschichte der Mathematik*, 1930). Although Sir Thomas Heath's manuscript was in the printer's hand before this publication, the general nature of the formula (to use a modern expression) for the volume of a truncated pyramid with a square base, as there given in Problem 14, was known (*Quellen und Studien* cited above, I, p. 87) and is here (p. 520) discussed. Just how this formula $V=\frac{1}{3}h(a^2+ab+b^2)$, a and b being sides of the square bases) was found is a problem awaiting solution. Could it have been derived empirically, or was it the result of rigid demonstration, and if so, upon what preliminary propositions was this based? Moreover, in the Moscow Papyrus (Problem 10) the formula for the surface of a hemisphere is correctly given, assuming the accuracy of the Egyptian value of π(*i.e.*, $\pi=256/81$), a piece of mensuration not mastered by the Greeks until the time of Archimedes. Rather prophetically Sir Thomas remarks (p. 522), "It may be that, even when the papyrus has been published, these puzzles will still remain unsolved." Indeed, as regards the problem of the truncated pyramid above mentioned, Dr. Kurt Vogel (*Journal of Egyptian Archaeology*, xvi, p. 249, Nov. 1930, and hence after the work under review appeared), after a careful study of the subject, concludes that "We have two alternatives . . . . We must allow to the Egyptians, side by side with a highly developed art of experiment, the arithmetical-algebraical equipment necessary to such a derivation [of the formula] in its full extent, [or] on the other hand if my suggested explanation be accepted we are no longer compelled to attribute to the Egyptians constructions and mental processes outside their ken." At the present moment it may well seem that the latter hypothesis is the safer.

Another source of light with respect to the predecessors of the Greeks is referred to on pp. 522 seq., namely, the recent publications of material relating to Babylonian mathematics. Surprising as are the two problems mentioned as being in the Moscow Papyrus, this material is even more so, and the questions which it raises will profoundly affect the decision as to the Greek claims of priority. The cuneiform tablets studied within the past three or four years confirm the belief that the Babylonian geometry was confined to mensuration, and this of a type requiring intuition rather than demonstration in the development of the necessary rules. What is more important is that the solutions apparently follow certain general rules which we express by formulas, and the process amounts to the solution of equations of considerable difficulty, although the work is wholly numerical and no rules are definitely set forth. For example, Neugebauer (*Quellen und Studien* cited above, I. Bd., 1. Heft., pp. 67–74, 1929) shows that one of the problems involves the solution of what we would now represent by the equations

$$\begin{aligned} u \cdot \tfrac{1}{2}(x + y) &= 783 \\ v \cdot \tfrac{1}{2}(y + z) &= 1377 \\ u &= \tfrac{1}{3}v \\ x - z &= (x - y) + (y - z) = 36 \\ x - y &= \tfrac{1}{3}(y - z) \end{aligned}$$

the roots being $x = 48$, $y = 39$, $z = 12$, $u = 18$, $v = 54$.

The same writer has also shown (p. 80) that a certain problem in one of the cuneiform texts in the British Museum leads to our equation

$$x^2 - \frac{2F}{d}\,x + \frac{2Fh}{d} = 0,$$

the root being calculated according to the formula or rule

$$x = \frac{F}{d} - \sqrt{\left(\frac{F}{d}\right)^2 - \frac{2Fh}{d}}.$$

This and further work of the same nature is set forth on the pages cited. If the interpretation of this new material is properly confirmed, the bearing of these discoveries upon the work of Euclid and Diophantus is significant.

A third item of importance, not found in the author's two-volume *History*, refers (p. 530) to recent discoveries showing that some of the records of the Babylonian astronomical observations, beginning in the reign of Nabonassar (747–734 B.C.), came into possession of the Greeks and were actually sent to Aristotle (d. 322 B.C.) after the capture of Babylon by Alexander (331 B.C.). These seem to have been used by Callippus (*fl.* c. 325 B.C.), and hence by Hypsicles, Hipparchus, Ptolemy, and other astronomers of the later Greek period. Not only is this important with respect to the close of the pre-Christian era,

but it also tends to confirm the common belief that Thales (c. 600) had access to similar records. (See also p. 89.)

The reference to the commentary of Pappus on Euclid X as "partly extant in Arabic" (p. 241) will, of course, need changing in view of the elaborate work of Junge and Thomson (*The Commentary of Pappus on Book X of Euclid's Elements*, Arabic text and translation, Harvard University Press, 1930), which appeared while this work was in press.

Such discoveries and studies give hope that definite information may sometime be forthcoming with respect to the first use of the *myriad* (Greek *myrias*, 10,000), the counting by 10,000's being oriental as well as Greek. Was there a common source in the Sumerian civilization? In speaking of Greek numeration some readers may regret that the work should give the impression that the numerals in the classical period were always represented by small letters instead of capitals. This often gives to beginners in the study of the history of notation a false idea of the facts. Another source of misunderstanding occurs on p. 25, where a medieval writer is quoted as saying that "the 'mensa Pythagorica' was so called in honor of the Master who taught its use." Since the quotation seems, at first sight at least, to be approved, these same beginners feel that it is worthy of credence. It is something like the confusion arising from the names "Naperian logarithms," logarithms which Napier never used, and "Arabic numerals" with which the Arabs were almost wholly unacquainted.

Altogether, the work is by far the best condensed presentation of the history of Greek mathematics that we have in any language.

DAVID EUGENE SMITH

College Algebra (*Alternate Edition*). By William L. Hart. D. C. Heath & Co., New York, 1931. 380 pages.

It would be interesting to know how many new editions of mathematical text-books are due to a burning desire on the part of the authors to improve the originals, and how many are inspired by the publishers' wish to eliminate used copies and teachers' suspicions that home-made keys to the problems have become too numerous. The title of this book implies that it belongs in the second category and, if this inference is correct, mathematical teachers should hope that the frankness of the author and publisher will be imitated by others, and that in the future no good book will be marred by attempts to make the new edition appear due to the first rather than to the second motive. This is certainly not true of the book under review, whatever the motives of the author; the old edition was a good book and the new one is better.

As to the difference between the two editions it is stated in the preface that "the original edition was rewritten in detail, without alteration of its main features." While this statement is literally true, nevertheless there is so little change in the text that a reader could shift from one to the other without noticing the difference. The principal alterations are a change in the order of four elementary chapters and a combination of two later chapters into one. The most

noticeable other changes are the omission of the numbering of steps in demonstrations, which does not seem to be an improvement, and the rewording, or rearrangement of the wording, of many sections in the interest of greater clarity. In contrast to the text there is an almost entirely new set of problems and their number has been increased from about 3,000 to 4,500.

For the benefit of those not acquainted with the old edition, it may be stated that Chapters I–VIII (pp. 1–111) cover algebra up to quadratics; Chapters IX–XVII (pp. 112–199) contain quadratics, ratio and proportion, variation, progressions, mathematical induction, the binomial theorem, and inequalities; and Chapters XVIII–XXVI (pp. 200–368) contain complex numbers, the theory of equations, logarithms, permutations and combinations, probability, determinants, partial fractions, the mathematics of investment, and infinite series. This is followed by seven pages of tables, an index, and answers to the odd-numbered problems. The object in using over half the book for high-school algebra is to provide a suitable starting place for the freshman with any degree of lack of preparation, an additional aid being the insertion of collections of review problems at the ends of Chapters IV, VI, and VIII.

The strong features of the book are the wealth of material, the clearness of presentation, and the attractive typography and arrangement; less desirable features are the lack of really difficult problems like the "stickers" in Hall and Knight and the encouragement to teacher and pupil to omit demonstrations, which raise a doubt that the student will gain enough power to comprehend the one really difficult chapter, that on determinants. However, such a criticism loses force when it is remembered that the book is for freshmen and therefore the emphasis is primarily not on rigorous thinking, but on "doing" algebra. In conclusion, the reviewer feels safe in predicting a successful career for this book in spite of his belief that the problem of more algebra for college students is better solved by letting the student get his algebra review in trigonometry, analytic geometry, and calculus, and by presenting the advanced topics to those who wish them is a more mature course later.

W. A. Wilson

Grundlehren der neueren Zahlentheorie, von Professor Dr. Paul Bachmann, Dritte, neu durchgesehene Auflage, herausgegeben von Professor Dr. Robert Haussner. Walter de Gruyter & Co., Berlin and Leipzig, 1931. xvi+252 pages. RM 9.50.

This third edition of Bachmann's well known introduction to the elementary theory of numbers from one fairly modern point of view, first published in 1907, differs from previous editions only in minor changes and in the following respects. Haussner has prefixed a short biography of his friend Bachmann, (1837–1920); Bachmann has added Frobenius' presentation of Zeller's proof of the law of quadratic reciprocity and a table classifying the 56 proofs of the law given up to 1914. Of the 56, 32 are by the Gauss lemma or variations of it, 8 by cyclotomy, 2 by quadratic fields.

The book can be warmly recommended to beginners with a good reading knowledge of German. The exposition is unusually detailed, and Bachmann spares no pains to make clear his reasons for attacking various arithmetical problems in the way he does. Opinions may differ whether a beginner should read some such book as this as his initiation to the theory of numbers, or whether he should start with something closer to the classical tradition. One who stops with the theory of integers and ideals for a quadratic field, as this book does, may get the too optimistic impression that the theory of numbers is rather an easy subject, and that all of its interesting problems yield their secrets to the methods of the theory of algebraic numbers. Such optimism should be corrected by further reading.

The first 143 pages cover the elementary theories of divisibility, congruence, quadratic residues, binary linear forms, and binary quadratic forms. Those who like geometrical pictures of algebraic or arithmetical facts will find several in this presentation of classical material. The second part of the book, pp. 144–252, is devoted to the quadratic field, and gives a detailed discussion of quadratic integers, modules and ideals, units of a quadratic field, divisibility in a quadratic field, ideals and lattice numbers. In this part much of the matter of the first part is re-presented in the language of quadratic algebraic number fields. Any inquiring reader who is tempted to ask why the first part was presented at all, in view of the greater power and elegance of the second, may be invited to try his hand on the theory of binary forms of degree 4, by any method that occurs to him.

E. T. BELL

Logarithmetica Britannica. Being a Standard Table of Logarithms to Twenty Decimal Places. By Alexander John Thompson. Part V, Numbers 50000 to 60000 Issued by the Biometric Laboratory, University of London, to Commemorate the Tercentenary of Henry Briggs' Publication of the *Arithmetica Logarithmica*, 1624. Subscription Issue. Cambridge, The University Press, 1931.

This is the fifth part (the fourth not yet published) of this tremendous undertaking. It consists of twenty-place logarithms of numbers of five digits, accompanied by values of second and fourth differences. The project speaks for itself; it is sufficient to say that the result is all that is to be expected of any product of the Cambridge Press.

R. A. J.

Standard Four-Figure Mathematical Tables. By L. M. Milne-Thomson and L. J. Comrie. London, Macmillan, 1931. xvi+246 pages, $8\times10\frac{1}{2}$ inches. $4.50.

This monumental work is issued in two editions, the one with negative characteristics of logarithms where they occur, the other with all such logarithms increased by 10. All the usual tables of logarithms and trigonometric functions, and logarithms of the latter, are given at length. There are also hyperbolic and

exponential functions and their inverses, natural logarithms, Gudermannians, etc. There is a single page for the gamma function, and one for the probability integral, represented by the symbol *Erf x*. There is a collection or formulae; including, for example, 139 integrals, and 53 series.

For convenience in interpolation a table of proportional parts is published on a card approximately the size of the page, so that it can be used in connection with any part of the book.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3503. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The apex angles of two cones of revolution are α_1 and α_2; the cones intersect in an ellipse whose semi-axes are a and b; show that the angle β which the axes of the cones make with each other is given by the equation.

$$\cos\beta = [-\sin^2\tfrac{1}{2}\alpha_1 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_1]^{1/2}[\sin^2\tfrac{1}{2}\alpha_2 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_2]^{1/2} - [1 - (b/a)^2]\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2.$$

3504. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Two cones of revolution whose apex angles are α_1 and α_2 intersect in an ellipse whose semi-major and semi-minor axes are a and b; show that

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2$$

in which γ_1 and γ_2 are the angles which the axes of the cones make with the plane of the ellipse.

3505. *Proposed by J. Rosenbaum, Milford, Conn.*

Under what condition will the lines joining the vertices of a tetrahedron with the points of contact of the opposite faces and the insphere be concurrent?

3506. *Proposed by E. B. Escott, Oak Park, Illinois.*

Solve these n simultaneous equations with n unknowns:

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ x_n & x_1 & \cdots & x_{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ x_3 & x_4 & \cdots & x_1 \end{vmatrix} = a_1; \quad \begin{vmatrix} x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ x_4 & x_5 & \cdots & x_2 \end{vmatrix} = a_2; \cdots$$

$$\begin{vmatrix} x_n & x_1 & \cdots & x_{n-2} \\ x_{n-1} & x_n & \cdots & x_{n-3} \\ \cdot & \cdot & \cdots & \cdot \\ x_2 & x_3 & \cdots & x_n \end{vmatrix} = a_n.$$

3507. *Proposed by Charles K. Robbins, Purdue University.*

Find a solution of

$$\frac{\partial v}{\partial t} = A\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{dv}{\partial r}\right)$$

subject to these conditions:

(1) $\partial v/\partial t = B - C(\partial v/\partial r)$ if $r = R$ and $0 < t < \infty$;

(2) $v = B$ if $t = 0$ and $0 \leqq r \leqq R$;

(3) $v = 0$ if $t = 0$ and $R < r < \infty$;

where A, B, C, and R are positive real constants.

3508. *Proposed by the late Artemas Martin, Washington, D. C.*

The sides of a plane triangle are a, b, c. It is required to determine the radius of the circle circumscribing the escribed circles of this triangle.

See the *Annals of Mathematics*, March, 1894.

SOLUTIONS

196. (192) [1913, 223; 1919, 214]. *Proposed by Charles Macaulay, Chicago, Illinois.*

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all the b's in the same column; all the c's in the same column, etc.

Solution by H. L. Olson, Michigan State College

It is easily seen that if there are more than two letters, it is not possible to arrange *all* the combinations containing a given even number of letters in this way; for if a and b are two letters which are to be placed in the same column, evidently the combination ab can not be used.

Assume that there is, in all, an even number, $2n$, of letters; we first separate

them into two groups, each containing n letters, which can be done in $(2n)!/(n!)^2$ ways if we understand that interchanging the groups gives a different separation. If each combination is to contain $2k$ letters, we combine each combination of k letters from the first group with each combination, in turn, of k letters from the second group. Thus the total number of ways in which the letters can be combined according to the conditions of the problem is ${}_nC_k^2$, or $(n!)^2(k!)^{-2}[(n-k)!]^{-2}$ for each separation into two groups.

291 [1914, 122; 1931, 171]. *Proposed by Emma M. Gibson, Springfield, Missouri.*

The time of descent, down a rough inclined plane, of a spherical shell which contains a smooth solid sphere of the same material as itself is t_1. The time of descent, down the same plane, of a solid sphere of the same material and radius as the shell is t_2. Determine the thickness of the shell.

From Loudon's *Elementary Theory of Rigid Dynamics*, p. 188.

Solution by Ralph P. Agnew, Princeton University.

Let the spheres be supposed to roll a distance l down a plane making an angle α with the horizontal. Let R be the radius of the spheres, and let r (unknown) be the radius of the inner sphere of the first sphere. Let the spheres have density d and mass M. Let I_1 denote the moment of inertia of the spherical shell, and let I_2 denote the moment of inertia of the second sphere. Let $s=s(t)$ be the distance rolled at time t, and let $\theta=\theta(t)$ be the angular displacement at time t; then $s=R\theta$.

For each sphere, the force parallel to the inclined plane is $Mg \sin \alpha$; and for each sphere this force may be regarded as the sum of two forces F_1 and F_2, the first of which produces the linear acceleration of the sphere and the second of which produces the rotational acceleration. By fundamental formulae, we have

$$F_1 = M(d^2s/dt^2),\ F_2R = I_n(d^2\theta/dt^2),\ n = 1, 2,$$

where $n=1$ refers to the first sphere and $n=2$ to the second. Hence, since $F_1+F_2=Mg \sin \alpha$ and $s=R\theta$,

$$(M + I_nR^{-2})(d^2s/dt^2) = Mg \sin \alpha,\ n = 1, 2,$$

and after integrating twice and using the facts that $s=(ds/dt)=0$ when $t=0$ and $s=l$ when $t=t_n$, we obtain

$$(M + I_nR^{-2})l = \tfrac{1}{2}Mgt_n^2 \sin \alpha,\ n = 1, 2.$$

Eliminating l from these equations, we obtain

$$(MR^2 + I_1)t_2^2 = (MR^2 + I_2)t_1^2.$$

Since $M=(4/3)d\pi R^3$, $I_2=(8/15)d\pi R^5$, and $I_1=(8/15)d\pi(R^5-r^5)$, the preceding relation reduces to

$$r^5 = 7R^5(t_2^2 - t_1^2)/(2t_2^2).$$

The thickness x of the spherical shell is therefore the following linear function of R:

$$x = R - r = \{1 - [7(t_2^2 - t_1^2)/(2t_2^2)]^{1/5}\}R.$$

Note by the Editors: It is implicitly assumed in the solution that, in the first case, the radius of the inner sphere is equal to the inner radius of the shell.

309 [1915, 162; 1931, 228]. *Proposed by Jos. B. Reynolds.*

The tangent at one cusp of a vertical three-arched hypocycloid is horizontal and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

Solution by the Proposer

Let the Y-axis be vertical and the coordinates of a point P on the curve be

$$x = 2a\cos\theta + a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta.$$

Let the curve cut the X-axis at the cusp A and let B be the highest cusp. For these points $\theta = 0$ and $2\pi/3$ respectively. The normal at P to the arc $AP = s$ makes a clockwise angle $\frac{1}{2}\theta$ with the vertical. The radius of curvature is $R = 8a\sin(3\theta/2)$ and $ds = 4a\sin(3\theta/2)\,d\theta$. Let w be the weight of the particle and k the coefficient of friction. Then resolving forces along the tangent and normal respectively at P we have

$$w\sin\tfrac{1}{2}\theta - kS = -wg^{-1}v\,dv/ds, \tag{1}$$

and

$$S - w\cos\tfrac{1}{2}\theta = wg^{-1}v^2/R; \tag{2}$$

in which v is the velocity of the particle and S the reaction of the curve.

Elimination of S from these equations gives,

$$g(\sin\tfrac{1}{2}\theta - k\cos\tfrac{1}{2}\theta) = kv^2/R - v\,dv/ds. \tag{3}$$

Substitution of $8a\sin(3\theta/2)$ for R and $4a\sin(3\theta/2)\,d\theta$ for ds gives the equation

$$8ag(\sin\tfrac{1}{2}\theta - k\cos\tfrac{1}{2}\theta)\sin(3\theta/2)\,d\theta = kv^2\,d\theta - 2v\,dv. \tag{4}$$

Integration of this equation leads to

$$4ag\left[\frac{(2-k^2)\sin 2\theta - 3k\cos 2\theta}{k^2+4} - \sin\theta\right] = v^2 + Ce^{k\theta}, \tag{5}$$

in which C is the constant of integration. Now $v = 0$ for $\theta = 0$ and also for $\theta = 2\pi/3$. These two pairs of values in (5) lead to two equations from which C may be eliminated giving the required equation in k or

$$2\sqrt{3} - k = 2ke^{2\pi k/3}.$$

Also solved by W. H. Erskine.

313 [1915, 202; 1931, 228]. *Proposed by Clifford N. Mills.*

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1+ekgw/2)$.

Solution by L. M. Hoskins, Palo Alto, Calif.

The stress per unit area upon a cross-section distant x below the point of support is $kg(c-x)$, and the elongation per unit length at this point is $kg(c-x)/e$. The total elongation is

$$\frac{kg}{e}\int_0^c (c-x)dx = kgc^2/2e.$$

The length of the stretched wire is therefore $c(1+kgc/2e)$ instead of the value given in the statement of the problem.

Also solved by Paul Capron.

315 [1915, 309; 1931, 229]. *Proposed by H. S. Uhler.*

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

Solution by L. M. Hoskins, Palo Alto, Calif.

Since in each case gravity is the only force doing work upon the cylinder, its kinetic energy *in any given position* during the descent would have the same value in the two cases. For the sliding body the energy is wholly translational, while for the rolling body it is in part rotational. The energy of translation is $\frac{1}{2}mv^2$ and that of rotation $\frac{1}{2}I\omega^2$, m being the mass, I the moment of inertia about the geometrical axis, v the velocity of the mass-center, and ω the angular velocity. If r is the radius, $I=\frac{1}{2}mr^2$, and for the case of rolling $\omega=v/r$, so that $\frac{1}{2}I\omega^2=\frac{1}{4}mv^2$. The total kinetic energy of the rolling cylinder is therefore $\frac{3}{4}mv^2$.

If v_1 and v_2 denote values of v *for the same position of the body* in the two cases of sliding and rolling

$$\tfrac{1}{2}mv_1^2 = \tfrac{3}{4}mv_2^2, \text{ or } v_1/v_2 = \sqrt{(3/2)}.$$

Since this fixed ratio of velocities holds for every position, its reciprocal gives the ratio of the times of descent. That is,

$$\frac{\text{time of descent of sliding body}}{\text{time of descent of rolling body}} = \sqrt{(2/3)}.$$

This result is independent of the radii of the cylinder and track.

For a sphere, since $I=2mr^2/5$, the rotational energy in the case of rolling is $\frac{1}{5}mv^2$ and the total energy $7mv^2/10$, leading to $\sqrt{(5/7)}$ instead of $\sqrt{(2/3)}$ as the ratio of the times of descent for the cases of sliding and rolling.

3464 [1930, 551]. *Proposed by I. Maizlish, Centenary College of Louisiana.*

A telegraph pole is in the form of a frustum of a right circular cone of altitude h, diameter of lower base D, and diameter of upper base d. A rope, of radius r, is wound spirally around this pole from the bottom to the top, covering the whole pole. A device is constructed which unwinds the rope—beginning at the top. If the angular velocity, w, with which the rope is being unwound is a function of the length of the rope already unwound, find the time it will take the device to unwind the whole rope.

Solution by H. R. Cooley, New York University

Let s be the length of the rope already unwound after it has turned through the angle θ. Then the time consumed by the complete unwinding is

$$t = \int_0^{\theta_1} \frac{d\theta}{f(s)}, \tag{1}$$

where $f(s)$ is the angular velocity and θ_1 is determined below in (2). Set $\frac{1}{2}(D-d)=a$, then the slant height of the frustum is $l=(a^2+h^2)^{1/2}$. If n is the number of turns of the rope necessary to cover the pole, then

$$2nr = l, \quad \theta_1 = 2\pi n = \pi l/r. \tag{2}$$

Consider a rectangular system of axes with the origin at the center of the upper base, the x-axis passing through the upper end of the rope, and with the z-axis downward along the axis of the frustum. Let ρ be the distance from the z-axis of a point of contact of the rope with the frustum. Then the equations of the curve of contact are

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = k\theta, \tag{3}$$

where k is a constant such that for $\theta=\theta_1$, $z=h$. From (2) $k=rh/l\pi$. From similar right triangles we have

$$\frac{\rho - \frac{1}{2}d}{a} = \frac{k\theta}{h}, \quad \text{or} \quad \rho = \frac{1}{2}d + \frac{ak\theta}{h}. \tag{4}$$

Hence from (3) and (4) we have

$$s = \int_0^{\theta} [(ak\theta/h + \tfrac{1}{2}d)^2 + r^2/\pi^2]^{1/2} d\theta;$$

and the integral on the right is easily evaluated by the formula for such integrals. With this value of s the time t is determined by (1).

A Note by the Editors: After reading the proof sheet of this solution, the solver has remarked that the expressions for z and l are not strictly correct, but for practical purposes, as in the case of an actual telegraph pole, the assumptions seem to be sufficiently accurate.

3467 [1930, 552]. *Proposed by J. Rosenbaum, Melford, Conn.*

Solve in positive integers: $2x^2+2x+1=y^2$.

Solution by H. C. Bradley, Massachusett Institute of Technology

This equation may be written: $x^2+(x+1)^2=y^2$. In this form, we see that the solution depends on that series of Pythagorean triangles whose perpendicular sides differ by unity, x being the value of the smaller side, and y the hypothenuse.

These triangles can be derived from the series of convergent fractions which result from extracting the square root of 2 by continued fractions. This series is

$$1/1,\ 3/2,\ 7/5,\ 17/12,\ 41/29,\ 99/70,\ 239/169,\ \cdots .$$

Each alternate fraction, beginning with the first, gives a solution, the numerator being $2x+1$, and the denominator y.

Simple as this is, it can be made still simpler. Write at once the fractions x/y in a series, thus:

$$0/1,\ 3/5,\ 20/29,\ 119/169,\ \cdots .$$

Then, let a/b and c/d be two consecutive fractions in this series; we note that $c=3a+2b+1$; $d=c+a+b+1$. By this law, the series may be continued as far as we please; the next two terms are 696/985 and 4059/5741.

Also solved by W. E. Buker, Mannis Charosh, Raymond Garver, Theodore Lindquist, R. E. Moritz, A. Pelletier, A. W. Randall, Wallace Smith, F. Underwood, E. E. Whitford, and the Proposer.

A Note by Otto Dunkel: The process above gives all the solutions; for the general theory of continued fractions for quadratic surds shows that any solution in positive integers of $2y^2-z^2=1$ must furnish a convergent z/y of the continued fraction development of $2^{1/2}$. If p_n/q_n is the nth convergent, it is also shown that

$$2q_n^2 - p_n^2 = (-1)^{n+1}, \tag{1}$$

and therefore the solution must give an odd convergent. Conversely an odd convergent furnishes a solution. From this last equation it is clear that p_n must be an odd integer, and hence it is always possible to find positive integral solutions for x_i, where $z=p_{2i+1}=2x_i+1$, $y_i=q_{2i+1}$. Then x_i, y_i is a solution of the equation of the problem, with $x_1=0$, $x_2=3$, $y_1=1$, $y_2=5$.

From the relations $p_{n+2}=2p_{n+1}+p_n$, $q_{n+2}=2q_{n+1}+q_n$, we easily derive $p_{n+2}=6p_n-p_{n-2}$ with a similar relation for the q's. From these result the equations

$$x_{i+2} = 6x_{i+1} - x_i + 2,\ \ y_{i+2} = 6y_{i+1} - y_i, \tag{2}$$

which enable us to calculate a solution from the two preceeding solutions. Other relations may be obtained from

$$p_n + 2^{1/2}q_n = \alpha^i(p_{n-i} + 2^{1/2}q_{n-i}),\ \ \alpha = 1 + 2^{1/2}, \tag{3}$$

by equating the rational and irrational parts of each side. Thus if $j=2$, $\alpha^2=3+2.2^{1/2}$, and

$$p_n = 3p_{n-2} + 4q_{n-2}, \quad q_n = 2p_{n-2} + 3q_{n-2};$$
$$x_{i+1} = 3x_i + 2y_i + 1, \quad y_{i+1} = 4x_i + 3y_i + 2. \tag{4}$$

This last pair of equations is essentially the same as the pair in the last part of the solution above. By means of (3) x_i and y_i may be expressed directly in terms of i, but the formulae thus obtained would not be as convenient for computation as the process given in the above solution.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1930–1931

Gamma of Pennsylvania, Lehigh University.

The officers for 1930–1931 were: Professor Tomlinson Fort, Director; John C. Mertz, Vice Director; John E. Freehafer, Second Vice Director; Harry C. Kelly, Secretary; Stephen L. Gregg, Treasurer; George J. Schaumburg, Librarian. The officers are elected by a majority vote near the end of the spring semester.

The meetings and programs were as follows:

October 24, 1930: "Foundations of geometry" by Professor Tomlinson Fort; "An introduction to non-Euclidean geometry" by Harry C. Kelly.

November 19, 1930: "A general introduction to the topic of projective geometry" by Professor Tomlinson Fort; "The principle of duality" by A. N. Rogers; "Cross ratio" by D. L. MacAdam.

December 15, 1930. "Second order configurations" by W. J. Tomlinson; "Complete quadrangle" by R. H. Raring.

February 16, 1931: "Theorems of Pascal and Brianchon" by W. C. Elmore; "Quadric surfaces" by P. G. Reynolds.

March 18, 1931: "Philosophical aspects of probability" by Benj. Rabinowitz; "Elementary theory of probability" by C. H. Krott; "Examples of probability using calculus methods" by J. B. Hartman.

April 22, 1931: "Humanism in mathematics and science" by Professor C. J. Keyser, Columbia University (This lecture was open to the public).

It is the policy of this chapter to concentrate on one subject of study, developing the topic in greater detail at successive meetings. Each spring there is a lecture open to the public.

This chapter has 45 active members. Twenty new members were initiated. The new members are tapped at a regular chapel exercise sometime in April and initiated in May. A banquet is held in connection with the initiation.

H. C. KELLY, *Secretary*

Alpha of California, University of California at Los Angeles.

California Alpha of Pi Mu Epsilon extends greetings and best wishes to its fraternal chapters. A very successful year has been enjoyed by the chapter under the direction of the following officers: Sibyl Rock, Director; Virginia Woods, Vice Director; Alta Blackford, Secretary; John Hill, Treasurer; Dr. Whyburn, Librarian; Dr. Daus, Faculty Adviser.

The meetings and programs were as follows:

October 8, 1930: Election to membership of Marjorie Easterly; Reed Lawlor; Abram Loshokoff; Jean Robb.

November 5, 1930: "Brocard points" by Annie Peterson.

December 3, 1930: "Inversion" by Francis Herrmenn.

December 6, 1930: Initiation of the newly elected members at the home of Alta Blackford.

December 28, 1930: Christmas party in conjunction with the Engineers Club and the Mathematics Club.

February 11, 1931: Dr. Slaught of the University of Chicago was made an honorary member of our chapter. At this time he gave a very interesting talk on "The growth and development of mathematical organizations." Dr. Slaught is our first honorary member.

March 9, 1931: Election to membership of Margaret Barney; Goldie Ivener; Chester Lagenbeck; John Montgomery.

March 25, 1931: "Subjects of tangents" by Marjorie Easterly; "Bohr theory" by Ernest Von Seggern.

April 18, 1931: Initiation of the newly elected members at the Mona Lisa cafe.

April 20, 1931: Special business meeting for the revision of rules and regulations concerning elegibility for membership.

May 1, 1931: Beach party at the Deauville Beach Club in conjunction with the Mathematics Club.

May 6, 1931: "Annual calculus examination." The award of ten dollars was given to Russell Doescher. Honorable mention was given to Donald Hyers.

May 13, 1931: "Definition of algebra" by Sibyl Rock. (Election of officers.)

ALTA BLACKFORD, *Secretary*

Pi Mu Epsilon of Syracuse University.

The officers for 1930–1931 were: Mr. David MacAlpine (Instructor in the College of Applied Science), Director; Sara Fister, Vice Director; Helen E. Kelley, Secretary; W. E. Moulton, Treasurer.

This chapter has a Scholarship Committee the members of which are: Professor I. S. Carroll, Chairman; Professors Taylor and Keenan; Sara Fister; Ruth Kohman; Malcolm Adler; Tilroe Hedden.

The meetings and programs were as follows:

October 7, 1930: Picnic. Major students in mathematics were the guests of the Chapter.

October 24, 1930: "Puzzles" by Dr. Hurwitz of Cornell University.

December 3, 1930: "Conic sections" by Mr. John Randolph. Twenty two students were elected to membership at this meeting. Of these, eight were graduate students, fourteen were undergraduates.

January 15, 1931: Initiation.

February 19, 1931: "Fourth dimension" by Dr. A. D. Campbell.

March 26, 1931: "The mathematics of aerial photography" by Professor Earl Church.

April 22, 1931: Business meeting.

May 6, 1931: Picnic. Major students in mathematics were the guests of the Chapter.

May 13, 1931: Initiation and Banquet. Forty-five members were present. Sara Fister was chairman of the banquet committee. Election of officers took place at the business meeting after the banquet.

MAY SPERRY, *Corresponding Secretary*

Pi Mu Epsilon of the University of Illinois.

The officers for 1930–1931 were: Mr. Marion T. Bird, Director; Mr. Donald M. Brown, Vice Director; Mr. Harry E. Crull, Corresponding Secretary; Miss Elsie Zelle, Recording Secretary; Mr. Julian K. Knipp, Treasurer.

Mr. Brown was elected October 14, 1930 to fill a vacancy. The other officers were elected at the regular spring election, May 20, 1930. All officers were elected by vote of the members present at these meetings.

The University of Illinois Chapter has 55 active members, of whom 30 were members at the beginning of the academic year, 14 were initiated December 16, 1930 and 11 were initiated May 4, 1931.

The meetings and programs were as follows:

October 28, 1930: "Analysis situs" by Professor Brahana.

January 13, 1931: "Space of many dimensions" by Dr. Wilson.

February 24, 1931: "An original theorem concerning the separation of the zeros of a function and its derivative" by Mr. F. S. Wood.

March 10, 1931: "Reverberation equation applied to a room" by Professor Watson. (This meeting was an open meeting.)

March 24, 1931: "Functions similar to the trigonometric functions" by Dr. O. K. Bower.

March 31, 1931: "A method of numerical integration of a first order differential equation" by Mr. Crull.

April 7, 1931: "Some remarks about the circle" by Professor Levy.

April 21, 1931: "Inversive invariants of three points" by Dr. Peters. (This meeting was an open meeting.)

On December 16, 1930 and May 4, 1931 we had our initiation banquets at the Inman Hotel in Champaign and on February 13, 1931, we had a theatre party at the Rialto Theatre in Champaign.

HARRY E. CRULL, *Corresponding Secretary*

Alpha of Alabama, University of Alabama.

The officers for 1930–1931 were: Charles E. Watkins, Director; Joseph S. Gelders, Vice Director; William F. Adams, Secretary; Brent G. Clark, Treasurer; Edith Gregory, Librarian.

We have 36 active members. An initiation was held on December 2, 1930 at which time 16 were initiated.

The meetings and programs were as follows:

September 30, 1930: "Exploring the house of arithmetic" by William Sell.

October 28, 1930: "Problem involving the determination of the orbit of one binary star around another having given the law of force" by Sara E. Haughton.

December 2, 1930: "To find the envelope of all the circles that cut two circles at a fixed angle" by Brent G. Clark.

January 27, 1931: "The Lorentz transformation" by Dr. B. A. Wooten.

February 24, 1931: "Two theorems of characteristic equations of square matrices" by H. S. Thurston.

April 7, 1931. "The foundations of mathematics" by Dr. W. P. Ott.

Alpha of Alabama has enjoyed two social meetings.

WILLIAM F. ADAMS, *Secretary*

Pi Mu Epsilon of the University of Pennsylvania.

This chapter of Pi Mu Epsilon has had a most successful year under the direction of the following officers. Dr. Maurice J. Babb, Director; Dr. George Gailey Chambers, Vice Director; Mr. Paul A. Knedler, Treasurer; Miss Margaret E. Koons, Secretary; Mr. Weinstein, Librarian.

The meetings and programs were as follows:

October 17, 1930: "Rational right triangles of equal areas" by Mr. W. A. Bristol.

November 21, 1930: "Mayan mathematics" by Miss Helen O'Boyle.

January 16, 1931: "The counting of molecules" by Dr. Horace C. Richards. (Department of Physics.)

February 20, 1931: "The cat's cradle and string figures" by Dr. H. M. Lufkin. (Department of Physics.)

March 20, 1931: "Infinitely continued exponents" by Mr. R. P. Bailey.

April 17, 1931: "Applications of mathematics to chemistry" by Mr. J. C. Miller. (Department of Chemistry.) At this meeting, 10 new members were initiated.

Prizes have been offered for the solution of interesting problems in mathematics and so far two excellent problems have been suggested and successfully solved by undergraduate students.

The May meeting is an open meeting. At this meeting the address is given by an outside speaker. We also have our annual picnic in May and this event is always well attended.

MARGARET E. KOONS, *Secretary*

Beta of Ohio, Ohio Wesleyan University.

The officers for the year 1930–1931 were: Raymond F. Felts, Director; Dean Friedly, Vice Director; Adelene Offinger, Secretary; John F. Foster, Treasurer.

The meetings and programs were as follows:

October 8, 1930: "Polygons and polyhedrons" by Professor Rufus Crane.

October 29, 1930: Business meeting. Election of new members.

November 11, 1930: Initiation. The club was entertained at the home of Professor and Mrs. Rowland.

December 8, 1930: "Education conference" by Miss Marie Gugle, Assistant Superintendent of Schools, Columbus, Ohio.

January 14, 1931: "Pythagoras and his theorem" by Mr. Paul Mathews.

February 11, 1931: "History and development of logarithms" by Mr. Wilbur Robinson.

March 9, 1931: "Problems of the student teacher" by five members. (Banquet.)

April 15, 1931: "Determinants" by Evelyn Coates.

May 13, 1931: "Statistical approach to the business depression" by Adelene Offinger.

ADELENE OFFINGER, *Secretary*

Delta of Pennsylvania, Pennsylvania State College.

The Pennsylvania Delta Chapter of Pi Mu Epsilon at present has 52 active members. Seven are charter members, 12 were elected at the beginning of the fall semester, 24 during that term and 9 during the current semester.

The officers for the first semester elected in October at the first meeting of the Chapter were: Dr. Haskell B. Curry, Director; Dr. Orrin Frink, Vice Director; Miss Gladys Quigg, Treasurer; Dr. Aline Huke, Secretary; Mr. H. L. Van Velzer, Librarian.

The officers for the second semester were: Mr. Evan Johnson, Director; Mr. George Fisanick, Vice Director; Miss Gladys Quigg, Treasurer; Mr. Eugene M. Fry, Secretary; Dr. Haskell B. Curry, Librarian.

Meetings were held with reasonable regularity during the year. The speakers were for the most part members of the Graduate Mathematics Faculty. Two meetings, however, were conducted by students and one by Dr. Altar, Instructor in Physics.

The meetings and programs were as follows:

October 29, 1930: "The fundamental laws of algebra" by Dr. Curry.

December 4, 1930: "Eclipses" by Dr. Rupp.

January 14, 1931: "Indeterminate equations" by Dr. Frink.

February 12, 1931: "Chinese mathematics" by Mr. Herple.

March 13, 1931: "A topic in descriptive astronomy" by Mr. Fry.

March 27, 1931: "The structure of physical theories" by Dr. Altar.
May 7, 1931: "The solutions of equations by successive approximations" by Dr. Sheffer.
May 20, 1931: "The foundations of geometry" by Dr. Owens.

Since the organization has now become more firmly established at Penn State it is hoped that next year we will be able to broaden our program and bring in visiting lecturers from other institutions. It is believed that the year's activities have justified the establishment of the chapter and we hope that, with the carrying through of the new program, the fraternity may come to be a real asset to the campus.

EVAN JOHNSON, *Director*

B.

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the George Washington University.

The officers for the year 1930–1931 were: Professor F. E. Johnston, President; Albert Wertheimer, Secretary. The officers are elected annually at the first meeting of the academic year.

The aim of the club is to stimulate a creative interest in mathematics. Membership is open to all persons having a genuine interest in the subject. This year we had 25 members.

The meetings and programs were as follows:
October 13, 1930: "Measurements of relation" by Dr. Frank M. Weida.
October 27, 1930: "The controversy over the foundations of mathematics" by Captain E. E. Hagler, Jr.
November 10, 1930: "Curve tracing" by Dr. F. E. Johnston.
November 24, 1930: "The history of non-Euclidean geometry" by Dr. Tobias Dantzig.
December 8, 1930: "Magic squares" by Abraham Sinkov.
January 12, 1931: "Ovals" by P. J. Federico.
February 25, 1931: "Maximum polyhedra" by Michael Goldberg.
March 11, 1931: "Applications of least squares" by Dr. W. E. Deming.
March 25, 1931: "Isometric projection in analytic geometry" by Dr. Walter F. Shenton.
April 15, 1931: "Hypercomplex variables" by Dr. Edgar W. Woolard.
April 29, 1931: "The reality of the imaginary" by Katherine G. Hawley.
May 16, 1931: "Some problems on the foundations of geometry" by Dr. Wilhelm Blaschke of Hamburg, Germany.

ALBERT WERTHEIMER, *Secretary*

The Undergraduate Mathematics Club of the University of Iowa.

The officers for the year 1930–1931 were: Carl H. Fischer, President; Thelma T. Coate, Secretary-Treasurer; Professor L. E. Ward, Faculty Advisor.

The meetings and programs were as follows:
October 29, 1930: "A recent problem in the American Mathematical Monthly" by Professor Roscoe Woods.
November 20, 1930: "The Galois theory of equations" by Mr. Allen T. Craig.
February 19, 1931: "The positive integers" by Mr. Deane Montgomery.
March 12, 1931: "An elementary method of computing logarithms" by Professor L. E. Ward.
April 16, 1931: "Contemporary American Mathematicians of the last thirty years" by Professor H. L. Rietz. At this meeting the officers for the year 1931–1932 were elected.

THELMA COATE, *Secretary*

The Newtonian Society of Lehigh University.

The officers for the year 1930–1931 were: Richard Lindabury, President; John B. Hancock, Vice President; David G. Wright, Secretary-Treasurer. The officers are elected immediately after the initiation of new members in February, the third meeting of the second semester.

The purpose of the society is to promote interest in mathematics among members of the

freshman class, to give its members opportunity for intellectual activity outside the classroom, and to promote friendship between its members and the members of the faculty.

Membership is an honor conferred only upon students of high standing in mathematics. Members are chosen from the freshman class at the end of each semester. The membership is limited to 40. The candidates for membership are nominated by their teachers and elected by a majority vote of the members present at a regular meeting. At present, we have 33 active members.

The meetings and programs were as follows:

February 23, 1931: "Life of Sir Isaac Newton" by J. Ricards.

March 9, 1931: "An old Hindu solution of the general quadratic" by R. Byers; "The slide rule and its inherent errors" by B. Beach.

March 23, 1931: "Paradoxes" by B. Fortman; "Outline of the history of mathematics" by K. Honeyman.

April 13, 1931: "Proof of the functions of the sum of two angles" by F. Geiger.

April 27, 1931: "The intersection of two parabolas" by C. MacDonald; "Derivation of logarithms" by P. Loughran; "Determination of three numbers, the sum of squares of two of which are equal to square of third" by Dr. E. H. Cutler.

May 11, 1931: "Universal utility of mathematical training" by Professor John Stocker. After this meeting the society held a social hour and smoker.

DAVID G. WRIGHT, *Secretary*

The Mathematics Club of Brown University.

The Mathematics Club at Brown University is run upon as informal a basis as possible. Beyond the necessary membership in the committees for the program and arranging the details of the meetings, there are no student officers. All undergraduates beyond the freshman year who are taking courses in mathematics are eligible for membership without other formality than the payment of $1.00 annual dues. The dues are used to cover the cost of refreshments at meetings. The student speakers are thoroughly drilled by members of the faculty, and the programs have been of sufficient interest to attract from 80 to 90 persons to most of the meetings. The program is printed and distributed in October, and no instance has occurred of incomplete preparation or absence of the announced speaker. Following each program there is a brief social hour at which on a few occasions there has been informal dancing. The students from Pembroke College share equally with the undergraduate men from Brown University.

The meetings and programs were as follows:

November 4, 1930, Wilson Hall, Edward M. Read, 3rd, '31, presiding: "Diophantus and Fermat" by Ruth Barden Eddy, '32; "Rational triangles" by Delbert S. Wicks, Jr., '32.

December 2, 1930, Pembroke Hall, Enis E. DeMagistris, '31, presiding: "The meaning of the mean" by James B. Brown, '31; "Prisms and pyramids" by Howard W. Memmott, '33.

January 13, 1931, Wilson Hall, Professor Currier, presiding "How Mother Earth keeps tab on Father Time" by Charles H. Smiley, Assistant Professor of Mathematics, Brown University.

February 24, 1931, Wilson Hall, Donald L. Fowler, Jr., '31, presiding: "Consequences of cutting a cone" by Tina Codianni, '33; "Seeing curves in the dark" by Raymond C. Archibald, Professor of Mathematics, Brown University.

March 24, 1931, Pembroke Hall, Ethel McKechnie, Gr., presiding: "Chow's tail" by William Solomon Wilson, '31; "Endless decimals and fractions" by Marjorie H. Smith, '32. (The club picture was taken at this meeting.)

April 28, 1931, Wilson Hall, Professor Archibald, presiding: "Symmetrical space forms," Illustrated by models, by Albert Harry Wheeler, North High School, Worcester, Massachusetts.

Committee on Program: Professor Bennett; Professor Oakley; Enis Eva DeMagistris, '31; Marjorie Helen Smith, '32; William Solomon Wilson, '31.

Committee on Arrangements: Mr. Jonah; Emma Mae Breyer, '31; Elizabeth Alma Partridge, '33; Austin Hazen, Jr., '32; Delbert Swan Wicks, Jr., '32.

ALBERT A. BENNETT, *Faculty Supervisor of the Program Committee*

The Junior Mathematics Club of the University of Chicago.

The Junior Mathematics Club of the University of Chicago is conducted by the graduate students of the department of mathematics. The officers of the Junior Club are elected by vote of the Club at the end of each academic year. The officers for 1930–1931 were Mr. Arnold E. Ross, President; Mr. Saunders MacLane, Secretary-Treasurer; Miss Ruth Mason, Social Committee.

The subjects presented at the meetings of the Club are usually of general mathematical interest. That is, they are historical sketches, surveys of certain domains of mathematics, or of applied mathematics, or discussions of famous classical problems. Any student in mathematics is eligible to membership. There are about fifty members.

In the current academic year the President and the Secretary of the Club edited the "Bulletin of the Junior Mathematics Club." The Bulletin consisted of a few typewritten pages, and contained an abstract of a forthcoming lecture, report of happenings in mathematics and among mathematicians, and other topics of interest to the members of the club. A new issue was placed on the bulletin board on alternate Mondays preceding the meetings.

The meetings and programs were as follows

October 8, 1930: "Historical Sketch of the Junior Mathematics Club" by Professor H. E. Slaught; "The Mathematical Association of America and the American Mathematical Society" by Professor M. I. Logsdon.

October 22, 1930: "An historical sketch of the theories of the calculus of variations" by Professor G. A. Bliss.

November 5, 1930: "Newton polygons in factorization of polynomials" by Mr. Saunders MacLane.

November 19, 1930: "The three pile match problem" by Professor W. D. MacMillan.

December 4, 1930: Trip to Adler Planetarium.

January 7, 1931: "Probabilities" by Professor Walter Bartky.

January 21, 1931: "The isoperimetric property of the circle" by Mr. Max Coral.

February 4, 1931: "Mathematical theories of life assurance" by Miss Frances Wiancko.

February 18, 1931: "Approximating plane curves by Newton polygons" by Rev. J. E. Case, S. J.

March 4, 1931: "Graphical methods for the determination of the elements of orbits of double stars" by Professor Kurt Laves.

April 10, 1931: "A revision of the theory of curvature" by Professor J. A. Schouten of Delft, Holland.

April 22, 1931: "An analogue of continued fractions" by Mr. M. R. Hestenes.

May 6, 1931: "A method for constructing cipher messages" by Dr. Mina Rees.

May 20, 1931: "Problems in mathematical economics" by Professor Henry Schultz.

June 3, 1931: "Life of Galois" by Miss Ruth Mason.

The Junior Club largely sponsors the social activities of the Department. A social half hour is held in the Common Room of Eckhart Hall before each Club meeting. Tea is served. During this year the Club has sponsored two dances and several bridge parties. One outstanding social evening in which all members of the Mathematics Department are included is always held some time during the year. The activities of the Junior Mathematics Club are quite distinct from and in addition to the work of the graduate research club which holds bi-weekly meetings.

ARNOLD ROSS, *President*

The University of Colorado Mathematics Club.

The object of this club shall be to stimulate among the members a spirit of inquiring interest in mathematics which will reveal to them more perfectly the cultural value of that science, and to promote a spirit of co-operation and friendship.

Any student enrolled in the University of Colorado who has completed successfully in a University or College of reputable standing three hours of mathematics is eligible to active membership. This year we had 21 active members.

The officers for 1930–1931 were: Mr. Robert A. Merrill, President; Mr. Marvin Halldorson, Vice President; Miss Janet Hall, Secretary-Treasurer; Professor A. J. Kempner, Faculty Advisor.

The meetings and programs were as follows:

October 8, 1930: Reading and ratification of the constitution.
October 16, 1930: "Fallacies in elementary mathematics" by Professor A. J. Kempner.
October 29, 1930: "Mathematical problems and their solutions" by Mr. Richard Furr.
November 13, 1930: "Fourier series" by Professor C. A. Hutchinson.
December 2, 1930: "Relations of time and space in the theory of relativity" by Mr. H. T. James.
January 8, 1931: "Hyperbolic functions and their use in engineering" by Mr. W. A. Wildhack.
January 22, 1931: "Crazy functions" by Mr. Jack Britton.
February 19, 1931: "Quadrature of the circle" by Miss Doris Hyddleston.
March 5, 1931: "Two theorems in geometry" by Miss Elizabeth Cole.
April 23, 1931: "Probability" by Mr. Robert A. Merrill.
May 14, 1931: "The history of calculus" by Miss Letha Lyon.
May 26, 1931: Math. Fry.
May 28, 1931: "Majoring in mathematics" by Professor G. H. Light.

ROBERT A. MERRILL, *President*

The Bryn Mawr Mathematics Club.

The officers for 1930–1931 were: Gretchen Mueller, '32, President; Ruth Unangst, '31, Vice President and Treasurer; Pauline Huger, '32, Secretary.

The officers were elected at the last meeting of the club last year on May 13, 1930. The club has eight active undergraduate members, three faculty members, and five honorary graduate members. The primary aim of the Mathematics Club is to stimulate interest in mathematics, and to give the students, who are taking advanced courses, a chance to hear interesting topics reported on and discussed. Only those students who are taking at least the major course are eligible for membership.

The meetings and programs were as follows:

December 2, 1930: "Some aspects of number theory" by Dr. Widder.
January 12, 1931: "Non-Euclidean geometry" by Pauline Huger.
February 2, 1931: "Geometry of compasses" by Dr. Hedlund.
April 29, 1931: "The Waring problem" by Gretchen Mueller.

There will be another meeting on May 23, to which all those students are invited who are planning to take the major course in mathematics. Dr. Lehr will speak, and after her talk the club plans to go on a picnic, where baseball and singing will most probably be indulged in.

PAULINE HUGER, *Secretary*

The Mathematics Klub of Adelphi College.

The officers for 1930–1931 were: Dr. Joseph Bowden, Honorary President; Lois Corbe, '31, President; Kate Sandowsky, '31, Vice President; Theresa Cartereau, '32, Secretary; Sarah Gordon, '32, Treasurer.

The officers are elected in May of the preceding semester by a two-thirds vote by ballot of the members present.

The object of the Mathematics Klub is to promote interest in the study of mathematics. Any person may be elected to membership in the Klub who is interested in the purpose of the organization. We have twenty-five active members.

The meetings and programs were as follows:

September 30, 1930: Annual party to welcome new members. Mathematical problems were presented by Kate Sandowsky, '31; Sarah Gordon, '32; Marie Garlichs, '31; Lois Corbe, '31.
October 8, 1930: "Congruences" by Dr. Joseph Bowden.
November 6, 1930: "The distance to the moon" and "Saturn's rings" by the President of the Astronomy Club. Observations were held in the College Observatory. Joint supper meeting of Mathematics and Astronomy Clubs.

November 24, 1930: "Trisection of an angle" by Adele Shrage, '32; "Gottfield von Leibnitz" by Kate Sandowsky, '31; "Harriot" by Sarah Gordon, '32.
Some mathematical problems were presented by various members of the klub.
December 2, 1930: "Scientific Calvinism" or "Predetermination" by Professor John A. David.
December 16, 1930: "Rhythm" by Professor William A. Thayer.
February 16, 1931: Professor Joseph Bowden and Lois Corbe, '31, gave proofs of interesting problems.
March 3, 1931: "Relation of mathematics to thoughts of God" by Dr. Ernest N. Henderson.
March 12, 1931: Supper meeting. A social meeting preceded the supper.
April 21, 1931: "Historical notes on the theory of the slide rule" by Theresa Cartereau, '32; "Demonstration of the slide rule" by Lois Corbe, '31.
May 5, 1931: Supper meeting. A play "Discord in Mathematics Land" was presented by the members. "Suggestions for future teachers" by Miss Ada Ostrander, Head of Mathematics Department, Sewanhaka High School, Floral Park, N. Y.
May 12, 1931: Nomination of officers for 1931–1932.
May 18, 1931: Election of officers for 1931–1932.
May 19, 1931: Installation of officers for 1931–1932.

THERESA CARTEREAU, '32, *Secretary*

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Eugenio Rignano Prize

The Review "Scientia," in agreement with the family of its late illustrious Director Eugenio Rignano, who died in Milan on February 9th., 1930, has founded a "Rignano Prize," of the value of 10,000 Lire, to be awarded by competition to an author of a study on "The Evolution of the Notion of Time." The Review believes that in taking this step it is paying the best tribute to the memory of its late Director by giving a new impulse to the scientific and philosophical studies to which he devoted his whole life, and by developing at the same time that union and spiritual emulation between the scientists of all countries, which was his highest aspiration.

The competition is completely international and open to all. Anyone can take part in it, without regard to school, tendency, or faculty.

Conditions of the Competition:

1.—The aspirants for the prize shall make known their intention of competing by sending their works to the Editor of "Scientia" not later than the 31st. of December, 1932.

2.—The works submitted must either be unpublished, or published since the year 1930, and must be written in one of the following languages: Italian, French, English, German, or Spanish. Works submitted must be typewritten or printed.

3.—In all cases, each work must be accompanied by a typewritten summary of not more than 10 pages (4000 words) in length, capable of being published as an article in the Review.

4.—The examination of the works submitted will be entrusted to a Committee nominated by the Directorate of the Review.

For all further information apply to the Editor of "Scientia," 12, Via A. De Togni, Milano (116), Italy.

Professor Henri Fehr, Head of the Department of Mathematics at the University of Geneva, has been appointed Recteur of the University. He is well-known to Americans and to the mathematical and educational world at large as the editor of l'Enseignement Mathématique, and as Secretary of the International Commission on the Teaching of Mathematics, not only the present one, but the one that was founded in 1908. He was a student at the Sorbonne and at the Collège de France. He has held a professorship in the University of Geneva since 1900, and was twice Dean of the Faculty of Science. From 1928 to 1930, he was Vice-Recteur of the University, and has frequently represented it at various ceremonies, at Oslo, Paris, Brussels and elsewhere. He was one of the founders and was president of the Société Mathématique Suisse. He is also a member of the Légion d'Honneur, besides being corresponding member of various mathematical societies.

Through the courtesy of Professor W. F. Shenton of the George Washington University, the Library of the Mathematical Association now possesses a complete set of the *Mathematical Visitor* published by Artemas Martin during the years 1877–1894, and of the *Mathematical Magazine* published by him during the years 1882–1910. This includes a copy of the very rare first edition of Volume 1, Number 1 of the *Mathematical Visitor*.

Todhunter's *History of the Theory of Probabilities* is being reprinted by G. E. Stechert & Co. This book has been so rare that it has been valued in the second hand market at from $40 to $50. But even at that price no copy could be found from which to make the reprint. It was necessary to borrow a copy from a university library. The price of the photographic print, equal to the original, will be $7.50.

Jenaro Moreno Garcia-Conde, Professor of Mathematics in the School of Arts and Crafts, School of Military Engineers and in the Military Staff College, Santiago, Chile, has been granted a fellowship by the John Simon Guggenheim Memorial Foundation for studies, in the United States, of the theory of functions of real variables and of complex variables, and of the calculus of variations of simple and multiple integrals.

M. Albert Linton was elected in June 1931 to the presidency of the Provident Mutual Life Insurance Company of Philadelphia. He entered the service of the Provident Mutual in 1909, became mathematician in 1913, associate actuary in 1915, and vice-president in 1916. He is a charter member of the Mathematical Association, fellow and vice-president of the Actuarial Society of America, fellow of the Institute of Actuaries of London, and chairman of the Life Insurance Sales Research Bureau of Hartford.

The Carus Mathematical Monographs

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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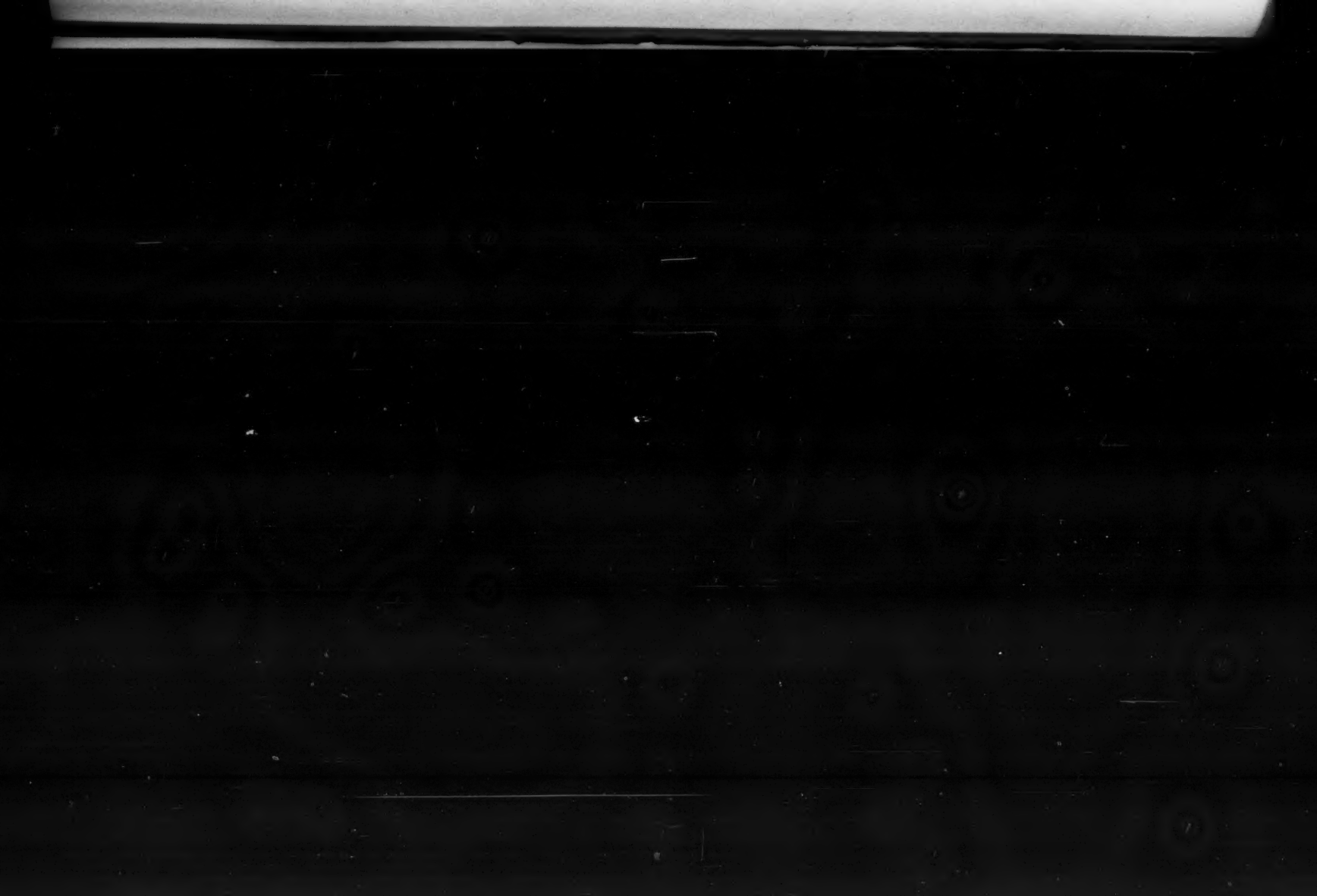
MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.



THE FIFTEENTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The fifteenth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colo., on Friday and Saturday, April 17 and 18, 1931. There were three sessions, Professor Claribel Kendall presiding at each.

The attendance was forty-eight, including the following twenty-three members of the Association: C. F. Barr, Jack Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, G. W. Gorrell, Sidney Hacker, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, O. C. Lester, A. J. Lewis, G. H. Light, S. L. Macdonald, A. S. McMaster, J. Q. McNatt, W. K. Nelson, Greta Neubauer, E. J. Purcell, E. D. Rainville, O. H. Rechard, Mary S. Sabin, C. H. Sisam.

At the business session the following officers were elected for the next year: Chairman, Professor O. H. Rechard, University of Wyoming; Vice-chairman, Professor A. G. Clark, Colorado Agricultural College. Plans were made to meet at the University of Wyoming in the spring of 1932.

The following papers were presented:

1. "On a biological application of the Poisson series" by Professor A. G. Clark, Colorado Agricultural College.
2. "A theorem on foci" by Mr. E. J. Purcell, University of Colorado.
3. "Projective geometry and some of its relations to other courses in mathematics and mathematical physics" by Professor J. R. Everett, Colorado School of Mines.
4. "Solution of a problem in dynamics" by Professor D. F. Gunder, Colorado Agricultural College, by invitation.
5. "On Riccati equations" by Professor C. A. Hutchinson, University of Colorado.
6. "Geometry as an avocation" by Professor A. J. Kempner, University of Colorado.
7. "The practical experiences of engineering in the mountain districts of Colorado" by Mr. J. Q. McNatt, Colorado Fuel and Iron Company.
8. "Linear systems of curves on an algebraic surface" by Professor C. H. Sisam, Colorado College.
9. "On the application of equations to the solution of congruences with prime moduli" by Mr. E. D. Rainville, University of Colorado.

Abstracts of some of the papers follow, the numbers corresponding to the numbers of the titles:

1. Professor Clark showed how the Poisson distribution could be used to simplify the work of the seed analyst in handling the problem of noxious weed seeds in certification work.

2. Mr. Purcell presented and proved the following theorem on foci, which he believes to be new: If a real algebraic curve of class M has one or more real